

# A theory of tensor products for module categories for a vertex operator algebra, IV\*

Yi-Zhi Huang

## Abstract

This is the fourth part of a series of papers developing a tensor product theory of modules for a vertex operator algebra. In this paper, We establish the associativity of  $P(z)$ -tensor products for nonzero complex numbers  $z$  constructed in Part III of the present series under suitable conditions. The associativity isomorphisms constructed in this paper are analogous to associativity isomorphisms for vector space tensor products in the sense that it relates the tensor products of three elements in three modules taken in different ways. The main new feature is that they are controlled by the decompositions of certain spheres with four punctures into spheres with three punctures using a sewing operation. We also show that under certain conditions, the existence of the associativity isomorphisms is equivalent to the associativity (or (nonmeromorphic) operator product expansion in the language of physicists) for the intertwining operators (or chiral vertex operators). Thus the associativity of tensor products provides a means to establish the (nonmeromorphic) operator product expansion.

The present paper (Part IV) is the fourth in a series of papers developing a theory of tensor products of modules for a vertex operator algebra. An

---

\*1991 *Mathematics Subject Classification*. Primary 17B69; Secondary 18D10, 81T40.

overview of the theory being developed has been given in [HL5] and the reader is referred to it for the motivation and the description of the main results.

In Part I ([HL3]), the notions of  $P(z)$ - and  $Q(z)$ -tensor product for any nonzero complex number  $z$  are introduced and two constructions of a  $Q(z)$ -tensor product are given based on certain results proved in Part II ([HL4]). In Part III ([HL6]), the notion of  $P(z)$ -tensor product is discussed in the same way as in Section 4 of Part I for that of  $Q(z)$ -tensor product, and two constructions of a  $P(z)$ -tensor product are given using the results for the  $Q(z)$ -tensor product. Part III ([HL6]) also contains a brief description of the results in [HL3] and [HL4]. In the present paper, the associativity for  $P(\cdot)$ -tensor products is formulated and the associativity isomorphisms are constructed under certain assumptions on the vertex operator algebra and on the products or iterates of two intertwining operators for the vertex operator algebra. These assumptions are satisfied by familiar examples and will be discussed in separate papers on applications of the theory of tensor products developed in the present series of papers. We also show that when the vertex operator algebra is rational and products of two intertwining operators are convergent in a suitable region, the existence of the associativity isomorphisms is equivalent to the associativity (or (nonmeromorphic) operator product expansion in the language of physicists) for the intertwining operators (or chiral vertex operators). See Theorems 14.11, 16.3 and 16.5 for the precise statements of the main results of the present paper.

Our conventions in this paper is the same as those in [HL3], [HL4] and [HL6]. We also add  $y$  to our list of formal variables. The symbols  $z_1, z_2, \dots$  will also denote nonzero complex numbers. We fix a vertex operator algebra  $V$ . The numberings of sections, formulas, etc., continues those of Part I, Part II and Part III.

Part IV is organized as follows: The associativity isomorphisms are constructed in Section 14, based on certain assumptions and some lemmas. We also prove in Section 14 that the existence of the associativity isomorphisms is equivalent to the associativity of the intertwining operators. The lemmas used in Section 14 are proved in Section 15. In Section 16, we give some conditions and show that they imply the assumptions used in the construction of the associativity isomorphisms.

**Acknowledgments** The present paper is one of the papers resulted from a long-term project jointly with James Lepowsky developing a tensor product theory of modules for a vertex operator algebra. I would like to express my gratitude to him for collaborations and many discussions. This work has been supported in part by NSF grants DMS-9104519 and DMS-9301020 and by DIMACS, an NSF Science and Technology Center funded under contract STC-88-09648.

## 14 Associativity isomorphisms for $P(\cdot)$ -tensor products

In this section, we construct the associativity isomorphisms for  $P(\cdot)$ -tensor products. To discuss the associativity we need to consider the compositions of one  $P(z_1)$ -intertwining map and one  $P(z_2)$ -intertwining map for suitable nonzero complex numbers  $z_1$  and  $z_2$ . Since intertwining maps are maps from  $W_1 \otimes W_2$  to  $\overline{W}_3$ , not to  $W_3$ , we have to give a precise meaning of the compositions of one  $P(z_1)$ -intertwining maps and one  $P(z_2)$ -intertwining map. Compositions of maps of this type have been defined precisely in [HL1] and [HL2]. Here we repeat the definition for the concrete examples of intertwining maps. Let  $F_1$  and  $F_2$  be  $P(z_1)$ - and  $P(z_2)$ -intertwining maps of type  $\binom{W_4}{W_1 W_5}$  and  $\binom{W_5}{W_2 W_3}$ , respectively. If for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$  and  $w'_{(4)} \in W'_4$ , the series

$$\sum_{n \in \mathbb{C}} \langle w'_{(4)}, F_1(w_{(1)} \otimes P_n(F_2(w_{(2)} \otimes w_{(3)}))) \rangle_{W_4} \quad (14.1)$$

is absolutely convergent, where for any  $n \in \mathbb{C}$ ,  $P_n$  is the projection map from a module to its homogeneous subspace of weight  $n$ , we say that *the product of  $F_1$  and  $F_2$  exists* and we call the map from  $W_1 \otimes W_2 \otimes W_3$  to  $\overline{W}_4$  defined by the limits above the *product* of  $F_1$  and  $F_2$ . Similarly, let  $F_3$  and  $F_4$  be  $P(z_3)$ - and  $P(z_4)$ -intertwining maps of type  $\binom{W_5}{W_1 W_2}$  and  $\binom{W_4}{W_5 W_3}$ , respectively. If for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$  and  $w'_{(4)} \in W'_4$ , the series

$$\sum_{n \in \mathbb{C}} \langle w'_{(4)}, F_4(P_n(F_3(w_{(1)} \otimes w_{(2)})) \otimes w_{(3)}) \rangle_{W_4} \quad (14.2)$$

is absolutely convergent, we say that *the iterate of  $F_3$  and  $F_4$  exists* and we call the map from  $W_1 \otimes W_2 \otimes W_3$  to  $\overline{W}_4$  defined by the limits above the

iterate of  $F_3$  and  $F_4$ .

Recall that for any nonzero complex number  $z$  and any integer  $p$ , there is an isomorphism between the space of  $P(z)$ -intertwining maps and the space of intertwining operators of the same type (see [HL6]). Take  $p = 0$  and let  $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$  and  $\mathcal{Y}_4$  be the intertwining operators correspond to  $F_1, F_2, F_3$  and  $F_4$ , respectively. (In this paper, we shall always use the isomorphism between the space of  $P(z)$ -intertwining maps and the space of intertwining operators of the same type associated to  $p = 0$ .) Then for  $i = 1, 2, 3, 4$ , we have

$$F_i(\cdot \otimes \cdot) = \mathcal{Y}_i(\cdot, x) \cdot \Big|_{x^n = e^{n \log z_i}, n \in \mathbb{C}}. \quad (14.3)$$

The absolute convergence of the series (14.1) and (14.2) are equivalent to the absolute convergence of the series

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}} \quad (14.4)$$

and

$$\langle w'_{(4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_1) w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1^n = e^{n \log z_3}, x_2^n = e^{n \log z_4}, n \in \mathbb{C}}, \quad (14.5)$$

respectively. For convenience, we shall write the substitutions, say,

$$\Big|_{x_1^n = e^{n \log z_1}, x_2^n = e^{n \log z_2}, n \in \mathbb{C}}$$

simply as  $\Big|_{x_1 = z_1, x_2 = z_2}$ .

Let  $V$  be a vertex operator algebra such that for any  $V$ -modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , any two nonzero complex numbers  $z_1$  and  $z_2$  satisfying  $|z_1| > |z_2| > 0$  and any  $P(z_1)$ -intertwining map  $F_1$  and  $P(z_2)$ -intertwining map  $F_2$  as above, (14.1) is absolutely convergent for all  $w_{(1)}, w_{(2)}, w_{(3)}$  and  $w'_{(4)}$ . Then (14.4) is also absolutely convergent. Recall that we have an isomorphism  $\Omega_{-1} : \mathcal{V}_{W_1 W_5}^{W_4} \rightarrow \mathcal{V}_{W_5 W_1}^{W_4}$  defined by

$$\Omega_{-1}(\mathcal{Y})(w_{(5)}, x) w_{(1)} = e^{xL(-1)} \mathcal{Y}(w_{(1)}, e^{-\pi i} x) w_{(5)}$$

for  $w_{(1)} \in W_1, w_{(5)} \in W_5$  and  $\mathcal{Y} \in \mathcal{V}_{W_1 W_5}^{W_4}$  (see [FHL] and [HL4]). Its inverse is  $\Omega_0 : \mathcal{V}_{W_5 W_1}^{W_4} \rightarrow \mathcal{V}_{W_1 W_5}^{W_4}$  defined by

$$\Omega_0(\mathcal{Y})(w_{(1)}, x) w_{(5)} = e^{xL(-1)} \mathcal{Y}(w_{(5)}, e^{\pi i} x) w_{(1)}$$

for  $w_{(1)} \in W_1$ ,  $w_{(5)} \in W_5$  and  $\mathcal{Y} \in \mathcal{V}_{W_5 W_1}^{W_4}$ . Let  $F_3$  and  $F_4$  be  $P(z_3)$ - and  $P(z_4)$ -intertwining maps, respectively, as above, and  $\mathcal{Y}_3$  and  $\mathcal{Y}_4$  the corresponding intertwining operators. Thus

$$\begin{aligned}
& \langle w'_{(4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_1)w_{(2)}, x_2)w_{(3)}) \rangle_{W_4} \Big|_{x_1=z_3, x_2=z_4} = \\
& = \langle w'_{(4)}, \Omega_0(\Omega_{-1}(\mathcal{Y}_4))(\mathcal{Y}_3(w_{(1)}, x_1)w_{(2)}, x_2)w_{(3)}) \rangle_{W_4} \Big|_{x_1=z_3, x_2=z_4} \\
& = \langle w'_{(4)}, e^{x_2 L^{(-1)}} \Omega_{-1}(\mathcal{Y}_4)(w_{(3)}, e^{\pi i} x_2) \mathcal{Y}_3(w_{(1)}, x_1)w_{(2)}) \rangle_{W_4} \Big|_{x_1=z_3, x_2=z_4} \\
& = \langle e^{x_2 L^{(1)}} w'_{(4)}, \Omega_{-1}(\mathcal{Y}_4)(w_{(3)}, e^{\pi i} x_2) \mathcal{Y}_3(w_{(1)}, x_1)w_{(2)}) \rangle_{W_4} \Big|_{x_1=z_3, x_2=z_4} \\
& = \langle e^{z_4 L^{(1)}} w'_{(4)}, \Omega_{-1}(\mathcal{Y}_4)(w_{(3)}, x_2) \mathcal{Y}_3(w_{(1)}, x_1)w_{(2)}) \rangle_{W_4} \Big|_{x_1=z_3, x_2=e^{\pi i} z_4},
\end{aligned} \tag{14.6}$$

where for convenience, we have used  $\Big|_{x_1=z_3, x_2=e^{\pi i} z_4}$  to denote

$$\Big|_{x_1^n=e^{n \log z_3}, x_2^n=e^{n \pi i} e^{n \log z_4}, n \in \mathbb{C}}$$

(and we shall use the similar notations below). Using the isomorphism between the space of  $P(-z_4)$ -intertwining maps and the space of intertwining operators and the isomorphism between the space of  $P(z_3)$ -intertwining maps and the space of intertwining operators, we see that the right-hand side of (14.6) is equal to the product of a  $P(-z_4)$ -intertwining map and a  $P(z_3)$ -intertwining map evaluated at  $w_{(3)} \otimes w_{(1)} \otimes w_{(2)} \in W_3 \otimes W_1 \otimes W_2$  and paired with  $e^{z_4 L^{(1)}} w'_{(4)} \in W'_4$ . By assumption, the right-hand side of (14.6) is convergent absolutely when  $|-z_4| > |z_3| > 0$  or equivalently when  $|z_4| > |z_3| > 0$ . Thus we have proved half of the following result:

**Proposition 14.1** *For any vertex operator algebra  $V$ , the following two properties are equivalent:*

1. *For any  $V$ -modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , any nonzero complex numbers  $z_1$  and  $z_2$  satisfying  $|z_1| > |z_2| > 0$  and any  $P(z_1)$ -intertwining map  $F_1$  of type  $\binom{W_4}{W_1 W_5}$  and  $P(z_2)$ -intertwining map  $F_2$  of type  $\binom{W_5}{W_2 W_3}$ , the product of  $F_1$  and  $F_2$  exists.*

2. For any  $V$ -modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , any nonzero complex numbers  $z_3$  and  $z_4$  satisfying  $|z_4| > |z_3| > 0$  and any  $P(z_3)$ -intertwining map  $F_3$  of type  $\begin{pmatrix} W_4 \\ W_5 W_3 \end{pmatrix}$  and  $P(z_4)$ -intertwining map  $F_4$  of type  $\begin{pmatrix} W_5 \\ W_1 W_2 \end{pmatrix}$ , the iterate of  $F_3$  and  $F_4$  exists.  $\square$

The other half of this result can be proved similarly.

We need the following result on products of the  $\delta$ -function:

**Lemma 14.2** *For any two nonzero complex numbers  $z_1$  and  $z_2$  satisfying*

$$|z_1| > |z_2| > |z_1 - z_2| > 0, \quad (14.7)$$

*we have*

$$\begin{aligned} x_1^{-1} \delta \left( \frac{x_0^{-1} - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) &= \\ &= x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right), \end{aligned} \quad (14.8)$$

$$\begin{aligned} z_1^{-1} \delta \left( \frac{x_0^{-1} - x_1}{z_1} \right) x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) &= \\ &= z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) (z_1 - z_2)^{-1} \delta \left( \frac{x_2 - x_1}{z_1 - z_2} \right), \end{aligned} \quad (14.9)$$

$$\begin{aligned} x_1^{-1} \delta \left( \frac{z_1 - x_0^{-1}}{x_1} \right) x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{x_2} \right) &= \\ &= x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right), \end{aligned} \quad (14.10)$$

$$\begin{aligned} x_1^{-1} \delta \left( \frac{z_1 - x_0^{-1}}{x_1} \right) z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) &= \\ &= z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) x_1^{-1} \delta \left( \frac{z_1 - z_2 - x_2}{x_1} \right). \end{aligned} \quad (14.11)$$

*The left-hand sides and the right-hand sides of these formulas are formal series in  $x_0, x_1, x_2$  whose coefficients converge absolutely when  $|z_1| > |z_2| > 0$  and when  $|z_2| > |z_1 - z_2| > 0$ , respectively, to rational functions of  $z_1$  and  $z_2$  with the only possible poles  $z_1 = \infty, 0$ ,  $z_2 = \infty, 0$  and  $z_1 = z_2$ .*

This result will be proved in the next section.

Now we assume that  $V$  satisfies either one of the properties in Proposition 14.1. By that proposition,  $V$  satisfies both properties. For any  $P(z_1)$ -intertwining map  $F_1$  and  $P(z_2)$ -intertwining map  $F_2$  as in Proposition 14.1, using the notation in the theory of (partial) operads (see [HL1] and [HL2]), we denote the product of  $F_1$  and  $F_2$  by  $\gamma(F_1; I, F_2)$ . The reader should note the well-definedness of the expressions in the calculations below. Using the definitions of  $\gamma(F_1; I, F_2)$  and of  $P(z_1)$ -intertwining map, we have

$$\begin{aligned}
& \langle w'_{(4)}, x_1^{-1} \delta \left( \frac{x_0 - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) Y_4(v, x_0) \cdot \\
& \quad \cdot \gamma(F_1; I, F_2)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle_{W_4} \\
&= \sum_{m \in \mathbb{C}} \langle w'_{(4)}, x_1^{-1} \delta \left( \frac{x_0 - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) \cdot \\
& \quad \cdot Y_4(v, x_0) P_m(\gamma(F_1; I, F_2)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})) \rangle_{W_4} \\
&= \sum_{m \in \mathbb{C}} \sum_{n \in \mathbb{C}} \langle w'_{(4)}, x_1^{-1} \delta \left( \frac{x_0 - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) \cdot \\
& \quad \cdot Y_4(v, x_0) P_m(F_1(w_{(1)}) \otimes P_n(F_2(w_{(2)} \otimes w_{(3)}))) \rangle_{W_4} \\
&= \sum_{m \in \mathbb{C}} \sum_{n \in \mathbb{C}} \langle w'_{(4)}, x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) P_m \left( x_1^{-1} \delta \left( \frac{x_0 - z_1}{x_1} \right) \cdot \right. \\
& \quad \cdot Y_4(v, x_0) F_1(w_{(1)} \otimes P_n(F_2(w_{(2)} \otimes w_{(3)}))) \rangle_{W_4} \\
&= \sum_{m \in \mathbb{C}} \sum_{n \in \mathbb{C}} \langle w'_{(4)}, x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) P_m \left( z_1^{-1} \delta \left( \frac{x_0 - x_1}{z_1} \right) \cdot \right. \\
& \quad \cdot F_1(Y_1(v, x_1) w_{(1)} \otimes P_n(F_2(w_{(2)} \otimes w_{(3)}))) \rangle_{W_4} \\
& \quad + \sum_{m \in \mathbb{C}} \sum_{n \in \mathbb{C}} \langle w'_{(4)}, x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) P_m \left( x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) \cdot \right. \\
& \quad \cdot F_1(w_{(1)} \otimes Y_5(v, x_0) P_n(F_2(w_{(2)} \otimes w_{(3)}))) \rangle_{W_4} \\
&= \langle w'_{(4)}, x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) z_1^{-1} \delta \left( \frac{x_0 - x_1}{z_1} \right) \cdot \\
& \quad \cdot \gamma(F_1; I, F_2)(Y_1(v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle_{W_4} \\
& \quad + \sum_{m \in \mathbb{C}} \sum_{n \in \mathbb{C}} \langle w'_{(4)}, x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) P_m \left( x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) \cdot \right.
\end{aligned}$$

$$\cdot F_1(w_{(1)} \otimes Y_5(v, x_0) P_n(F_2(w_{(2)} \otimes w_{(3)}))) \rangle_{W_4} \quad (14.12)$$

for all  $v \in V$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$  and  $w'_{(4)} \in W'_4$ . By the definition of  $P(z_2)$ -intertwining map, the second term in the right-hand side of (14.12) becomes

$$\begin{aligned} & \sum_{m \in \mathbb{C}} \sum_{n \in \mathbb{C}} \langle w'_{(4)}, P_m \left( x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) F_1 \left( w_{(1)} \otimes \right. \right. \\ & \quad \left. \left. \otimes P_n \left( x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) Y_5(v, x_0) F_2(w_{(2)} \otimes w_{(3)}) \right) \right) \right) \rangle_{W_4} \\ &= \sum_{m \in \mathbb{C}} \sum_{n \in \mathbb{C}} \langle w'_{(4)}, P_m \left( x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) F_1 \left( w_{(1)} \otimes \right. \right. \\ & \quad \left. \left. \otimes P_n \left( z_2^{-1} \delta \left( \frac{x_0 - x_2}{z_2} \right) F_2(Y_2(v, x_2) w_{(2)} \otimes w_{(3)}) \right) \right) \right) \rangle_{W_4} \\ & \quad + \sum_{m \in \mathbb{C}} \sum_{n \in \mathbb{C}} \langle w'_{(4)}, P_m \left( x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) F_1 \left( w_{(1)} \otimes \right. \right. \\ & \quad \left. \left. \otimes P_n \left( x_2^{-1} \delta \left( \frac{z_2 - x_0}{-x_2} \right) F_2(w_{(2)} \otimes Y_3(v, x_0) w_{(3)}) \right) \right) \right) \rangle_{W_4} \\ &= \langle w'_{(4)}, x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) z_2^{-1} \delta \left( \frac{x_0 - x_2}{z_2} \right) \cdot \\ & \quad \cdot \gamma(F_1; I, F_2)(w_{(1)} \otimes Y_2(v, x_2) w_{(2)} \otimes w_{(3)}) \rangle_{W_4} \\ & \quad + \langle w'_{(4)}, x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) x_2^{-1} \delta \left( \frac{z_2 - x_0}{-x_2} \right) \cdot \\ & \quad \cdot \gamma(F_1; I, F_2)(w_{(1)} \otimes w_{(2)} \otimes Y_3(v, x_0) w_{(3)}) \rangle_{W_4}. \end{aligned} \quad (14.13)$$

Substituting (14.13) into (14.12), we obtain

$$\begin{aligned} & \langle w'_{(4)}, x_1^{-1} \delta \left( \frac{x_0 - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) Y_4(v, x_0) \cdot \\ & \quad \cdot \gamma(F_1; I, F_2)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle_{W_4} \\ &= \langle w'_{(4)}, x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) z_1^{-1} \delta \left( \frac{x_0 - x_1}{z_1} \right) \cdot \\ & \quad \cdot \gamma(F_1; I, F_2)(Y_1(v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle_{W_4} \\ & \quad + \langle w'_{(4)}, x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) z_2^{-1} \delta \left( \frac{x_0 - x_2}{z_2} \right) \cdot \\ & \quad \cdot \gamma(F_1; I, F_2)(w_{(1)} \otimes Y_2(v, x_2) w_{(2)} \otimes w_{(3)}) \rangle_{W_4} \end{aligned}$$



$$\begin{aligned}
& + \langle w'_{(4)}, x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) x_2^{-1} \delta \left( \frac{z_2 - x_0}{-x_2} \right) \cdot \\
& \quad \cdot \gamma(F_1; I, F_2)(w_{(1)} \otimes w_{(2)} \otimes Y_3(v, x_0)w_{(3)}) \rangle_{W_4}. \tag{14.14}
\end{aligned}$$

Since (14.14) holds for any  $w'_{(4)} \in W'_4$ , we obtain

$$\begin{aligned}
& x_1^{-1} \delta \left( \frac{x_0 - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) \cdot \\
& \quad \cdot (Y_4(v, x_0) \gamma(F_1; I, F_2))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& = x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) z_1^{-1} \delta \left( \frac{x_0 - x_1}{z_1} \right) \cdot \\
& \quad \cdot \gamma(F_1; I, F_2)(Y_1(v, x_1)w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& \quad + x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) z_2^{-1} \delta \left( \frac{x_0 - x_2}{z_2} \right) \cdot \\
& \quad \cdot \gamma(F_1; I, F_2)(w_{(1)} \otimes Y_2(v, x_2)w_{(2)} \otimes w_{(3)}) \\
& \quad + x_1^{-1} \delta \left( \frac{z_1 - x_0}{-x_1} \right) x_2^{-1} \delta \left( \frac{z_2 - x_0}{-x_2} \right) \cdot \\
& \quad \cdot \gamma(F_1; I, F_2)(w_{(1)} \otimes w_{(2)} \otimes Y_3(v, x_0)w_{(3)}). \tag{14.15}
\end{aligned}$$

We are mainly interested in (14.14). The left-hand side of (14.14) can be written as

$$\begin{aligned}
& \langle x_1^{-1} \delta \left( \frac{x_0 - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0 - z_2}{x_2} \right) \cdot \\
& \quad \cdot Y'_4(e^{x_0 L(1)} (-x_0^2)^{-L(0)} v, x_0^{-1}) w'_{(4)}, \gamma(F_1; I, F_2)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle_{W_4}. \tag{14.16}
\end{aligned}$$

First replacing  $v$  by  $(-x_0^2)^{L(0)} e^{-x_0 L(1)} v$  and then replacing  $x_0$  by  $x_0^{-1}$  in both (14.16) and the right-hand side of (14.14), we obtain

$$\begin{aligned}
& \langle x_1^{-1} \delta \left( \frac{x_0^{-1} - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \cdot \\
& \quad \cdot Y'_4(v, x_0) w'_{(4)}, \gamma(F_1; I, F_2)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle_{W_4} \\
& = \langle w'_{(4)}, x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) z_1^{-1} \delta \left( \frac{x_0^{-1} - x_1}{z_1} \right) \cdot \\
& \quad \cdot \gamma(F_1; I, F_2)(Y_1((-x_0^{-2})^{L(0)} e^{-x_0^{-1} L(1)} v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle_{W_4}
\end{aligned}$$

$$\begin{aligned}
& + \langle w'_{(4)}, x_1^{-1} \delta \left( \frac{z_1 - x_0^{-1}}{-x_1} \right) z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) \cdot \\
& \quad \cdot \gamma(F_1; I, F_2)(w_{(1)} \otimes Y_2((-x_0^{-2})^{L(0)} e^{-x_0^{-1}L(1)} v, x_2) w_{(2)} \otimes w_{(3)}) \rangle_{W_4} \\
& + \langle w'_{(4)}, x_1^{-1} \delta \left( \frac{z_1 - x_0^{-1}}{-x_1} \right) x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{-x_2} \right) \cdot \\
& \quad \cdot \gamma(F_1; I, F_2)(w_{(1)} \otimes w_{(2)} \otimes Y_3((-x_0^{-2})^{L(0)} e^{-x_0^{-1}L(1)} v, x_0^{-1}) w_{(3)}) \rangle_{W_4}.
\end{aligned} \tag{14.17}$$

Note that the map  $\gamma(F_1; I, F_2) : W_1 \otimes W_2 \otimes W_3 \rightarrow \overline{W}_4$  amounts to a map from  $W'_4$  to  $(W_1 \otimes W_2 \otimes W_3)^*$  and that the coefficients of

$$x_1^{-1} \delta \left( \frac{x_0^{-1} - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) Y_t(v, x_0)$$

in powers of  $x_0, x_1$  and  $x_2$ , for all  $v \in V$ , span

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}].$$

Thus the formula (14.17) motivates the definition of the following action  $\tau_{P(z_1, z_2)}^{(1)}$  of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}]$$

on  $(W_1 \otimes W_2 \otimes W_3)^*$ :

$$\begin{aligned}
& (\tau_{P(z_1, z_2)}^{(1)}(x_1^{-1} \delta \left( \frac{x_0^{-1} - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) Y_t(v, x_0)) \lambda)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& = x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) z_1^{-1} \delta \left( \frac{x_0^{-1} - x_1}{z_1} \right) \cdot \\
& \quad \cdot \lambda(Y_1((-x_0^{-2})^{L(0)} e^{-x_0^{-1}L(1)} v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& + x_1^{-1} \delta \left( \frac{z_1 - x_0^{-1}}{-x_1} \right) z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes Y_2((-x_0^{-2})^{L(0)} e^{-x_0^{-1}L(1)} v, x_2) w_{(2)} \otimes w_{(3)}) \\
& + x_1^{-1} \delta \left( \frac{z_1 - x_0^{-1}}{-x_1} \right) x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{-x_2} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes w_{(2)} \otimes Y_3((-x_0^{-2})^{L(0)} e^{-x_0^{-1}L(1)} v, x_0^{-1}) w_{(3)}) \tag{14.18}
\end{aligned}$$

for all  $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$  and  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ .

Similarly, for any  $P(z_3)$ -intertwining map  $F_3$  and  $P(z_4)$ -intertwining map  $F_4$  as in Proposition 14.1, we denote the iterate of  $F_3$  and  $F_4$  by  $\gamma(F_4; F_3, I)$ . Similarly to the discussion for  $\gamma(F_1; I, F_2)$ , we can show that

$$\begin{aligned}
& \langle x_2^{-1} \delta \left( \frac{x_0^{-1} - z_4}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - z_3}{x_1} \right) \cdot \\
& \quad \cdot Y'_4(v, x_0) w'_{(4)}, \gamma(F_4; F_3, I)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle_{W_4} \\
& = \langle w'_{(4)}, x_2^{-1} \delta \left( \frac{z_4 - x_0^{-1}}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - z_3}{x_1} \right) \cdot \\
& \quad \cdot \gamma(F_4; F_3, I)(w_{(1)} \otimes w_{(2)} \otimes Y((-x_0^{-2})^{L(0)} e^{-x_0^{-1} L(1)} v, x_0^{-1}) w_{(3)}) \rangle_{W_{(4)}} \\
& \quad + \langle w'_{(4)}, z_4^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_4} \right) z_3^{-1} \delta \left( \frac{x_2 - x_1}{z_3} \right) \cdot \\
& \quad \cdot \gamma(F_4; F_3, I)(Y((-x_0^{-2})^{L(0)} e^{-x_0^{-1} L(1)} v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \rangle_{W_{(4)}} \\
& \quad + \langle w'_{(4)}, z_4^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_4} \right) x_1^{-1} \delta \left( \frac{z_3 - x_2}{x_1} \right) \cdot \\
& \quad \cdot \gamma(F_4; F_3, I)(w_{(1)} \otimes Y((-x_0^{-2})^{L(0)} e^{-x_0^{-1} L(1)} v, x_2) w_{(2)} \otimes w_{(3)}) \rangle_{W_{(4)}}
\end{aligned} \tag{14.19}$$

for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$  and  $w'_{(4)} \in W'_4$ . Note that all terms in both sides of (14.19) are well defined. Since the components of

$$x_2^{-1} \delta \left( \frac{x_0^{-1} - z_4}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - z_3}{x_1} \right) Y_t(v, x_0)$$

in powers of  $x_0, x_1$  and  $x_2$  span

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, ((z_3 + z_4)^{-1} - t)^{-1}, (z_4^{-1} - t)^{-1}],$$

(14.19) motivates the following definition of an action  $\tau_{P(z_3+z_4, z_4)}^{(2)}$  of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, ((z_3 + z_4)^{-1} - t)^{-1}, (z_4^{-1} - t)^{-1}]$$

on  $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ :

$$\left( \tau_{P(z_3+z_4, z_4)}^{(2)} \left( x_2^{-1} \delta \left( \frac{x_0^{-1} - z_4}{x_2} \right) \right) \right).$$

$$\begin{aligned}
& \cdot x_1^{-1} \delta \left( \frac{x_2 - z_3}{x_1} \right) Y_t(v, x_0) \lambda \Big( w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \Big) \\
&= x_2^{-1} \delta \left( \frac{z_4 - x_0^{-1}}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - z_3}{x_1} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes w_{(2)} \otimes Y((-x_0^{-2})^{L(0)} e^{-x_0^{-1}L(1)} v, x_0^{-1}) w_{(3)}) \\
& + z_4^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_4} \right) z_3^{-1} \delta \left( \frac{x_2 - x_1}{z_3} \right) \cdot \\
& \quad \cdot \lambda(Y((-x_0^{-2})^{L(0)} e^{-x_0^{-1}L(1)} v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& + z_4^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_4} \right) x_1^{-1} \delta \left( \frac{z_3 - x_2}{x_1} \right) \cdot \\
& \quad \cdot \lambda(w_{(1)} \otimes Y((-x_0^{-2})^{L(0)} e^{-x_0^{-1}L(1)} v, x_2) w_{(2)} \otimes w_{(3)}) \quad (14.20)
\end{aligned}$$

for all  $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$  and  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ .

From (14.8), (14.9), (14.10) and (14.11), we see that

$$\tau_{P(z_1, z_2)}^{(1)} = \tau_{P(z_1, z_2)}^{(2)} \quad (14.21)$$

when (14.7) holds.

When (14.7) holds, let

$$L'_{P(z_1, z_2)}(0) = \tau_{P(z_1, z_2)}^{(1)}(\omega \otimes t^0) = \tau_{P(z_1, z_2)}^{(2)}(\omega \otimes t^0) \quad (14.22)$$

and

$$Y'_{P(z_1, z_2)}(v, x) = \tau_{P(z_1, z_2)}^{(1)}(Y_t(v, x)) = \tau_{P(z_1, z_2)}^{(2)}(Y_t(v, x)). \quad (14.23)$$

We call the eigenspaces of the operator  $L'_{P(z_1, z_2)}(0)$  the  $P(z_1, z_2)$ -weight subspaces (or  $P(z_1, z_2)$ -homogeneous subspaces) of  $(W_1 \otimes W_2 \otimes W_3)^*$ , and we have the corresponding notions of  $P(z_1, z_2)$ -weight vector (or  $P(z_1, z_2)$ -homogeneous vector) and  $P(z_1, z_2)$ -weight.

Let  $W_1, W_2, W_3, W_4$  and  $W_5$  be  $V$ -modules,  $F_1, F_2, F_3$  and  $F_4$   $P(z_1)$ -,  $P(z_2)$ -,  $P(z_1 - z_2)$ - and  $P(z_2)$ -intertwining maps of type  $\begin{pmatrix} W_4 \\ W_1 W_5 \end{pmatrix}$ ,  $\begin{pmatrix} W_5 \\ W_2 W_3 \end{pmatrix}$ ,  $\begin{pmatrix} W_5 \\ W_1 W_2 \end{pmatrix}$  and  $\begin{pmatrix} W_4 \\ W_5 W_3 \end{pmatrix}$ , respectively. Then the restrictions to  $W'_4$  of the adjoint maps of the product  $\gamma(F_1; I, F_2)$  of  $F_1$  and  $F_2$  when  $|z_1| > |z_2| > 0$  and of the iterate  $\gamma(F_4; F_3, I)$  of  $F_3$  and  $F_4$  when  $|z_2| > |z_1 - z_2| > 0$  are maps

$\gamma(F_1; I, F_2)'$  and  $\gamma(F_4; F_3, I)'$ , respectively, from  $W'_4$  to  $(W_1 \otimes W_2 \otimes W_3)^*$  defined by

$$\begin{aligned} & (\gamma(F_1; I, F_2)'(w'_{(4)}))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = \\ & = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \end{aligned} \quad (14.24)$$

and

$$\begin{aligned} & (\gamma(F_4; F_3, I)'(w'_{(4)}))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = \\ & = \langle w'_{(4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_1) w_{(2)}, x_2) w_{(3)}) \rangle_{W_4} \Big|_{x_1=z_3, x_2=z_4}, \end{aligned} \quad (14.25)$$

respectively. It is easy to verify that for any  $w'_{(4)} \in W'_4$ ,  $\gamma(F_1; I, F_2)'(w'_{(4)})$  and  $\gamma(F_3; F_4, I)'(w'_{(4)})$  satisfy the following conditions for  $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$  when (14.7) holds:

**The  $P(z_1, z_2)$ -compatibility condition**

(a) The  $P(z_1, z_2)$ -lower truncation condition: For all  $v \in V$ , the formal Laurent series  $Y'_{P(z_1, z_2)}(v, x)\lambda$  involves only finitely many negative powers of  $x$ .

(b) The following formula holds:

$$\begin{aligned} & \tau_{P(z_1, z_2)}^{(1)} \left( x_0^{-1} \delta \left( \frac{x_0^{-1} - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) Y_t(v, x_0) \right) \lambda = \\ & = x_1^{-1} \delta \left( \frac{x_0^{-1} - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \tau_{P(z_1, z_2)}^{(1)}(Y_t(v, x_0)) \lambda \\ & \text{for all } v \in V, \end{aligned} \quad (14.26)$$

or equivalently,

$$\begin{aligned} & \tau_{P(z_1, z_2)}^{(2)} \left( x_0^{-1} \delta \left( \frac{x_0^{-1} - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) Y_t(v, x_0) \right) \lambda = \\ & = x_1^{-1} \delta \left( \frac{x_0^{-1} - z_1}{x_1} \right) x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \tau_{P(z_1, z_2)}^{(2)}(Y_t(v, x_0)) \lambda \\ & \text{for all } v \in V. \end{aligned} \quad (14.27)$$

**The  $P(z_1, z_2)$ -local grading-restriction condition**

(a) The  $P(z_1, z_2)$ -grading condition:  $\lambda$  is a (finite) sum of eigenvectors of  $L'_{P(z_1, z_2)}(0)$ .

(b) Let  $W_\lambda$  be the smallest subspace of  $(W_1 \otimes W_2 \otimes W_3)^*$  containing  $\lambda$  and stable under the component operators  $\tau_{P(z_1, z_2)}^{(1)}(v \otimes t^n)$  of the operators  $Y'_{P(z_1, z_2)}(v, x)$  for  $v \in V$ ,  $n \in \mathbb{Z}$ . Then the weight spaces  $(W_\lambda)_{(n)}$ ,  $n \in \mathbb{C}$ , of the (graded) space  $W_\lambda$  have the properties

$$\dim (W_\lambda)_{(n)} < \infty \quad \text{for } n \in \mathbb{C}, \quad (14.28)$$

$$(W_\lambda)_{(n)} = 0 \quad \text{for } n \text{ whose real part is sufficiently small.} \quad (14.29)$$

These two conditions are the analogues of the corresponding conditions for  $\tau_{P(z)}$  and  $\tau_{Q(z)}$ . But in this case, we also have to consider two additional conditions. To state the conditions, it is convenient to introduce the following concepts: Let  $W$  be a vector space and  $W^*$  its dual. A formal sum  $\sum_{n \in \mathbb{C}} w_n^*$  where  $w_n^* \in W^*$ ,  $n \in \mathbb{C}$ , is called a *series* in  $W^*$ . A series  $\sum_{n \in \mathbb{C}} w_n^*$  in  $W^*$  is said to be *absolutely convergent* if for any  $w \in W$ , the sum  $\sum_{n \in \mathbb{C}} |w_n^*(w)|$  is convergent. If  $\sum_{n \in \mathbb{C}} w_n^*$  is absolutely convergent, then for any  $w \in W$ , the sum  $\sum_{n \in \mathbb{C}} w_n^*(w)$  is also convergent and the limits for all  $w \in W$  define an element of  $W^*$ . This element is called the *limit* of the series  $\sum_{n \in \mathbb{C}} w_n^*$ . If a series  $\sum_{n \in \mathbb{C}} w_n^*$  of homogeneous vectors in a  $\mathbb{C}$ -graded subspace of  $W^*$  have the property that for any  $w \in W$ , the weights of all those vectors  $w_n^*$  in the series such that  $w_n^*(w) \neq 0$  can be arranged to give a sequence of strictly increasing real numbers, we say that this series is a *series indexed by sequences of strictly increasing real numbers*. We consider the following two conditions for  $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ :

**The  $P(z_2)$ -local grading restriction condition**

(a) The  $P(z_2)$ -grading condition: For any  $w_{(1)} \in W_1$ , the element  $\mu_{\lambda, w_{(1)}}^{(1)} \in (W_2 \otimes W_3)^*$  defined by

$$\mu_{\lambda, w_{(1)}}^{(1)}(w_{(2)} \otimes w_{(3)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \quad (14.30)$$

for all  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ , is the limit of an absolutely convergent series of  $P(z_2)$ -weight vectors in  $(W_2 \otimes W_3)^*$  indexed by sequences of strictly increasing real numbers.

(b) For any  $w_{(1)} \in W_1$ , let  $W_{\lambda, w_{(1)}}^{(1)}$  be the smallest subspace of  $(W_2 \otimes W_3)^*$  containing the  $P(z_2)$ -weight vectors in the series absolutely convergent to  $\mu_{\lambda, w_{(1)}}^{(1)}$  and stable under the component operators  $\tau_{P(z_2)}(v \otimes t^n)$  of the operators  $Y'_{P(z_2)}(v, x)$  for  $v \in V$ ,  $n \in \mathbb{Z}$ . Then the weight spaces  $(W_{\lambda, w_{(1)}}^{(1)})_{(n)}$ ,  $n \in \mathbb{C}$ , of the (graded) space  $W_{\lambda, w_{(1)}}^{(1)}$  have the properties

$$\dim (W_{\lambda, w_{(1)}}^{(1)})_{(n)} < \infty \quad \text{for } n \in \mathbb{C}, \quad (14.31)$$

$$(W_{\lambda, w_{(1)}}^{(1)})_{(n)} = 0 \quad \text{for } n \text{ whose real part is sufficiently small.} \quad (14.32)$$

**The  $P(z_1 - z_2)$ -local grading restriction condition**

(a) The  $P(z_1 - z_2)$ -grading condition: For any  $w_{(3)} \in W_3$ , the element  $\mu_{\lambda, w_{(3)}}^{(2)} \in (W_1 \otimes W_2)^*$  defined by

$$\mu_{\lambda, w_{(3)}}^{(2)}(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \quad (14.33)$$

for all  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ , is the limit of an absolutely convergent series of  $P(z_1 - z_2)$ -weight vectors in  $(W_1 \otimes W_2)^*$  indexed by sequences of strictly increasing real numbers.

(b) For any  $w_{(3)} \in W_3$ , let  $W_{\lambda, w_{(3)}}^{(2)}$  be the smallest subspace of  $(W_1 \otimes W_2)^*$  containing the  $P(z_1 - z_2)$ -weight vectors in the series absolutely convergent to  $\mu_{\lambda, w_{(3)}}^{(2)}$  and stable under the component operators

$$\tau_{P(z_1 - z_2)}(v \otimes t^n)$$

of the operators  $Y'_{P(z_1 - z_2)}(v, x)$  for  $v \in V$ ,  $n \in \mathbb{Z}$ . Then the weight spaces  $(W_{\lambda, w_{(3)}}^{(2)})_{(n)}$ ,  $n \in \mathbb{C}$ , of the (graded) space  $W_{\lambda, w_{(3)}}^{(2)}$  have the properties

$$\dim (W_{\lambda, w_{(3)}}^{(2)})_{(n)} < \infty \quad \text{for } n \in \mathbb{C}, \quad (14.34)$$

$$(W_{\lambda, w_{(3)}}^{(2)})_{(n)} = 0 \quad \text{for } n \text{ whose real part is sufficiently small.} \quad (14.35)$$

We have:

**Lemma 14.3** *Let  $z_1, z_2$  be complex numbers satisfying (14.7). Assume that  $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$  satisfies the  $P(z_1, z_2)$ -compatibility condition. Then for any  $v \in V$  and  $w_{(3)} \in W_3$ ,*

$$x_1^{-1} \delta \left( \frac{x^{-1} - (z_1 - z_2)}{x_1} \right) Y'_{P(z_1 - z_2)}(v, x) \mu_{\lambda, w_{(3)}}^{(2)}$$

*is well defined and we have*

$$\begin{aligned} \tau_{P(z_1 - z_2)} \left( x_1^{-1} \delta \left( \frac{x^{-1} - (z_1 - z_2)}{x_1} \right) Y_t(v, x) \right) \mu_{\lambda, w_{(3)}}^{(2)} &= \\ &= x_1^{-1} \delta \left( \frac{x^{-1} - (z_1 - z_2)}{x_1} \right) Y'_{P(z_1 - z_2)}(v, x) \mu_{\lambda, w_{(3)}}^{(2)}. \end{aligned} \quad (14.36)$$

*Similarly, for any  $v \in V$  and any  $w_{(1)} \in W_1$ ,*

$$x_1^{-1} \delta \left( \frac{x^{-1} - z_2}{x_1} \right) Y'_{P(z_2)}(v, x) \mu_{\lambda, w_{(1)}}^{(1)}$$

*is well defined and we have*

$$\begin{aligned} \tau_{P(z_2)} \left( x_1^{-1} \delta \left( \frac{x^{-1} - z_2}{x_1} \right) Y_t(v, x) \right) \mu_{\lambda, w_{(1)}}^{(1)} &= \\ &= x_1^{-1} \delta \left( \frac{x^{-1} - z_2}{x_1} \right) Y'_{P(z_1 - z_2)}(v, x) \mu_{\lambda, w_{(1)}}^{(1)}. \end{aligned} \quad (14.37)$$

The proof of this result will be given in the next section.

We now assume that  $V$  is rational and all irreducible  $V$ -modules are  $\mathbb{R}$ -graded. Note that in this case all  $V$ -modules are  $\mathbb{R}$ -graded. We show that  $\gamma(F_1; I, F_2)'(w'_{(4)})$  satisfies the  $P(z_2)$ -local grading-restriction condition. We assume that  $w'_{(4)}$  is homogeneous. For any homogeneous element  $w_{(1)} \in W_1$ , let  $\alpha_n(w'_{(4)}, w_{(1)})$ ,  $n \in \mathbb{C}$ , be the elements of  $(W_2 \otimes W_3)^*$  defined by

$$(\alpha_n(w'_{(4)}, w_{(1)}))(w_{(2)} \otimes w_{(3)}) = \langle w'_{(4)}, F_1(w_{(1)} \otimes P_n(F_2(w_{(2)} \otimes w_{(3)}))) \rangle_{W_4}. \quad (14.38)$$

By the commutator formula for  $L(0)$  and  $F_2$ , the definition  $L'_{P(z_2)}(0)$  and the definition of intertwining map, it is easy to verify that  $\alpha_n(w'_{(4)}, w_{(1)})$ ,  $n \in \mathbb{C}$ , are  $P(z_2)$ -weight vectors and are in  $W_2 \boxtimes_{P(z_2)} W_3$ . From our assumption,



the series  $\sum_{n \in \mathbb{C}} \alpha_n(w'_{(4)}, w_{(1)})$  is absolutely convergent to  $\mu_{\gamma(F_1; I, F_2)'(w'_{(4)}, w_{(1)})}^{(1)}$ . To show that this series is indexed by sequences of strictly increasing real numbers, we need the following lemma, whose proof will be given in the next section:

**Lemma 14.4** *Let  $W_1$ ,  $W_2$  and  $W_3$  be three modules for a rational vertex operator algebra  $V$  and  $\mathcal{Y} : W_1 \otimes W_2 \rightarrow W_3\{x\}$  an intertwining operator of type  $\left( \begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix} \right)$ . Then there exist finitely many complex numbers  $h_1, \dots, h_p$  such that  $\mathcal{Y}$  is in fact a map from  $W_1 \otimes W_2$  to  $x^{h_1} W_3[[x, x^{-1}]] + \dots + x^{h_p} W_3[[x, x^{-1}]]$ . If all irreducible  $V$ -modules are  $\mathbb{R}$ -graded,  $h_1, \dots, h_p$  are real numbers.*

Combining this lemma with the lower truncation condition for the intertwining operator  $\mathcal{Y}_2$  corresponding to  $F_2$ , we see that there exists a sequence  $\{n_i\}_{i \in \mathbb{Z}_+}$  of strictly increasing real numbers such that

$$\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)} = \sum_{i \in \mathbb{Z}_+} (w_{(2)})_{-n_i-1} w_{(3)} x_2^{n_i}. \quad (14.39)$$

Thus for any  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ , if  $n \neq \text{wt } w_{(2)} + n_i + 1 + \text{wt } w_{(3)}$ ,  $i \in \mathbb{Z}_+$ ,

$$\begin{aligned} & (\alpha_n(w'_{(4)}, w_{(1)}))(w_{(2)} \otimes w_{(3)}) = \\ &= \langle w'_{(4)}, F_1(w_{(1)} \otimes P_n(F_2(w_{(2)} \otimes w_{(3)}))) \rangle_{W_4} \\ &= \langle w'_{(4)}, F_1(w_{(1)} \otimes P_n(\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)})) \rangle_{W_4} \Big|_{x_2=z_2} \\ &= \langle w'_{(4)}, F_1(w_{(1)} \otimes P_n(\sum_{i \in \mathbb{Z}_+} (w_{(2)})_{-n_i-1} w_{(3)} x_2^{n_i})) \rangle_{W_4} \Big|_{x_2=z_2} \\ &= 0. \end{aligned} \quad (14.40)$$

So  $\sum_{n \in \mathbb{C}} \alpha_n(w'_{(4)}, w_{(1)})$  is indexed by sequences of strictly increasing real numbers. This proves that  $\gamma(F_1; I, F_2)'(w'_{(4)})$  satisfies the  $P(z_2)$ -local grading-restriction condition. Similarly we can show that  $\gamma(F_4; F_3, I)'(w'_{(4)})$  satisfies the  $P(z_1 - z_2)$ -local grading restriction condition.

Assume in addition that the vertex operator algebra  $V$  has the property that for any modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , any  $z_1, z_2 \in \mathbb{C}$  satisfying (14.7), any  $P(z_1)$ - and  $P(z_2)$ -intertwining maps  $F_1$  and  $F_2$  of the types as above and any  $w'_{(4)} \in W'_4$ , the element  $\gamma(F_1; I, F_2)'(w'_{(4)})$  in  $(W_1 \otimes W_2 \otimes W_3)^*$

satisfies the  $P(z_1 - z_2)$ -local grading-restriction condition. Let  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  be the intertwining operators corresponding to  $F_1$  and  $F_2$ , respectively. For  $z \in \mathbb{C}^\times$ , we define the  $P(z_1 - z_2 + zz_2)$ - and  $P(zz_2)$ -intertwining maps  $F_1^z$  and  $F_2^z$  of the same types as those of  $F_1$  and  $F_2$ , respectively, by

$$\begin{aligned} F_1^z &= \mathcal{Y}(\cdot, x) \cdot \Big|_{x=z_1-z_2+zz_2}, \\ F_2^z &= \mathcal{Y}(\cdot, x) \cdot \Big|_{x=zz_2}. \end{aligned} \quad (14.41)$$

If  $0 < |z| < \frac{|z_1-z_2|}{2|z_2|}$ , we have  $|z_1 - z_2 + zz_2| > |zz_2| > 0$ . Thus the product  $\gamma(F_1^z; I, F_2^z)$  exists.

By assumption, there is a series  $\sum_{n \in \mathbb{C}} \alpha_n(w'_{(4)}, w_{(3)})$  indexed by sequences of strictly increasing real numbers of  $P(z_1 - z_2)$ -weight vectors in  $(W_1 \otimes W_2)^*$  converging absolutely to the element

$$\mu_{\gamma(F_1; I, F_2)'(w'_{(4)}), w_{(3)}}^{(2)} \quad (14.42)$$

of  $(W_1 \otimes W_2)^*$  when (14.7) holds. By the definition of (14.42),  $L'_{P(z_1-z_2)}(0)$  and  $\gamma(F_1; I, F_1)'$ , we see that for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w'_{(4)} \in W'_4$  and formal variable  $x$ ,

$$\begin{aligned} &((1-x)^{-L'_{P(z_1-z_2)}(0)} \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}), w_{(3)}}^{(2)})(w_{(1)} \otimes w_{(2)}) = \\ &= \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}), w_{(3)}}^{(2)}((1-x)^{-(z_1-z_2)L(-1)-L(0)} w_{(1)} \otimes (1-x)^{-L(0)} w_{(2)}) \\ &= (\gamma(F_1; I, F_1)'(w'_{(4)}))((1-x)^{-(z_1-z_2)L(-1)-L(0)} w_{(1)} \otimes \\ &\quad \otimes (1-x)^{-L(0)} w_{(2)} \otimes w_{(3)}) \\ &= \langle w'_{(4)}, \mathcal{Y}_1((1-x)^{-(z_1-z_2)L(-1)-L(0)} w_{(1)}, x_1) \cdot \\ &\quad \cdot \mathcal{Y}_2((1-x)^{-L(0)} w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2}. \end{aligned} \quad (14.43)$$

Using the commutator formulas for  $L(0)$ ,  $L(-1)$  and intertwining operators, it is easy to show by direct calculations that the right-hand side of (14.43) is equal to

$$\begin{aligned} &\langle w'_{(4)}, (1-x)^{-(L(0)-z_2L(-1))} \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\ &\quad \cdot \mathcal{Y}_2(w_{(2)}, x_2) (1-x)^{L(0)-z_2L(-1)} w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2}. \end{aligned} \quad (14.44)$$

By Lemma 9.3 in [HL4] which gives the formula

$$(1-x)^{L(0)-z_2L(-1)} = e^{z_2xL(-1)}(1-x)^{L(0)},$$

(14.44) is equal to

$$\begin{aligned} & \langle w'_{(4)}, (1-x)^{-L(0)} e^{-z_2xL(-1)} \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\ & \quad \cdot \mathcal{Y}_2(w_{(2)}, x_2) e^{z_2xL(-1)} (1-x)^{L(0)} w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\ &= \langle (1-x)^{-L(0)} w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1 - z_2x) \cdot \\ & \quad \cdot \mathcal{Y}_2(w_{(2)}, x_2 - z_2x) (1-x)^{L(0)} w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\ &= \langle (1-x)^{-L(0)} w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\ & \quad \cdot \mathcal{Y}_2(w_{(2)}, x_2) (1-x)^{L(0)} w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1-z_2x, x_2=z_2-z_2x}. \end{aligned} \quad (14.45)$$

The calculations above shows that the left-hand side of (14.43) is equal to the right-hand side of (14.45). Since in the right-hand side of (14.45), we can substitute  $1 - e^{\log z}$  for  $x$  when  $0 < |z| < \frac{|z_1-z_2|}{2|z_2|}$ , we can also substitute  $1 - e^{\log z}$  for  $x$  when  $0 < |z| < \frac{|z_1-z_2|}{2|z_2|}$  in the left-hand side of (14.43), and after the substitutions, the left-hand side of (14.43) and the right-hand side of (14.45) are still equal. So we have

$$\begin{aligned} & (e^{-(\log z)L'_{P(z_1-z_2)}(0)} \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}, w_{(3)})}^{(2)})(w_{(1)} \otimes w_{(2)}) = \\ &= \langle e^{-(\log z)L'(0)} w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\ & \quad \cdot \mathcal{Y}_2(w_{(2)}, x_2) e^{(\log z)L(0)} w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1-z_2+zz_2, x_2=zz_2} \\ &= (\mu_{\gamma(F_1^z; I, F_2^z)'(e^{-(\log z)L'(0)} w'_{(4)}, e^{(\log z)L(0)} w_{(3)})}^{(2)})(w_{(1)} \otimes w_{(2)}). \end{aligned} \quad (14.46)$$

Since  $\alpha_n(w'_{(4)}, w_{(3)})$ ,  $n \in \mathbb{C}$ , are weight vectors, we have

$$\begin{aligned} & e^{-(\log z)L'_{P(z_1-z_2)}(0)} \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}, w_{(3)})}^{(2)} = \\ &= \sum_{n \in \mathbb{C}} \alpha_n(w'_{(4)}, w_{(3)}) e^{-\text{wt } \alpha_n(w'_{(4)}, w_{(3)}) \log z} \end{aligned} \quad (14.47)$$

when (14.7) holds and  $0 < |z| < \frac{|z_1 - z_2|}{2|z_2|}$ . By (14.46) and (14.47), we obtain

$$\begin{aligned} \mu_{\gamma(F_1^z; I, F_2^z)'(w'_{(4)}, w_{(3)})}^{(2)} &= \\ &= \sum_{n \in \mathbb{C}} \alpha_n(w'_{(4)}, w_{(3)}) e^{-(\text{wt } \alpha_n(w'_{(4)}, w_{(3)}) + \text{wt } w'_{(4)} - \text{wt } w_{(3)}) \log z} \end{aligned} \quad (14.48)$$

when (14.7) holds and  $0 < |z| < \frac{|z_1 - z_2|}{2|z_2|}$ . When

$$|z_1 - z_2 + zz_2| > |zz_2| > |z_1 - z_2| > 0,$$

by Lemma 14.3,  $\mu_{\gamma(F_1^z; I, F_2^z)'(w'_{(4)}, w_{(3)})}^{(2)}$  satisfies (14.36). Since the open set on the  $z$ -plane given by  $|z_1 - z_2 + zz_2| > |zz_2| > 0$  is connected and  $\mu_{\gamma(F_1^z; I, F_2^z)'(w'_{(4)}, w_{(3)})}^{(2)}$  is defined and analytic in  $z$  when  $|z_1 - z_2 + zz_2| > |zz_2| > 0$ , the coefficients in powers of  $x$  and  $x_1$  of both sides of (14.36) with  $\lambda = \gamma(F_1^z; I, F_2^z)'(w'_{(4)})$ , as (multi-valued) functions of  $z$  can be analytically extended to  $|z_1 - z_2 + zz_2| > |zz_2| > 0$  and are still equal. When (14.7) holds and  $0 < |z| < \frac{|z_1 - z_2|}{2|z_2|}$ , these coefficients can be expanded as series in powers of  $e^{\log z}$ . By part (b) of the  $P(z_1 - z_2)$ -local grading-restriction condition which we assumed to be satisfied by  $\gamma(F_1; I, F_2)$ ,  $\alpha_n(w'_{(4)}, w_{(3)})$ ,  $n \in \mathbb{C}$ , satisfies the  $P(z_1 - z_2)$ -lower truncation condition. Thus for any  $n \in \mathbb{C}$ ,

$$x_1^{-1} \delta \left( \frac{x^{-1} - (z_1 - z_2)}{x_1} \right) Y'_{P(z_1 - z_2)}(v, x) \alpha_n(w'_{(4)}, w_{(3)})$$

exists. By (14.48), the coefficients in powers of  $x$  and  $x_1$  of

$$\begin{aligned} &x_1^{-1} \delta \left( \frac{x^{-1} - (z_1 - z_2)}{x_1} \right) Y'_{P(z_1 - z_2)}(v, x) \cdot \\ &\cdot \left( \sum_{n \in \mathbb{C}} \alpha_n(w'_{(4)}, w_{(3)}) e^{-(\text{wt } \alpha_n(w'_{(4)}, w_{(3)}) + \text{wt } w'_{(4)} - \text{wt } w_{(3)}) \log z} \right) \end{aligned}$$

exist since they are in fact the expansions as series in powers of  $e^{\log z}$  of the analytic extensions of the coefficients in powers of  $x$  and  $x_1$  of the right-hand side of (14.36) with  $\lambda = \gamma(F_1^z; I, F_2^z)'(w'_{(4)})$ . We obtain

$$\tau_{P(z_1 - z_2)} \left( x_1^{-1} \delta \left( \frac{x^{-1} - (z_1 - z_2)}{x_1} \right) Y_t(v, x) \right).$$

$$\begin{aligned}
& \cdot \left( \sum_{n \in \mathbb{C}} \alpha_n(w'_{(4)}, w_{(3)}) e^{-(\text{wt } \alpha_n(w'_{(4)}, w_{(3)}) + \text{wt } w'_{(4)} - \text{wt } w_{(3)}) \log z} \right) \\
&= x_1^{-1} \delta \left( \frac{x^{-1} - (z_1 - z_2)}{x_1} \right) Y'_{P(z_1 - z_2)}(v, x) \cdot \\
& \cdot \left( \sum_{n \in \mathbb{C}} \alpha_n(w'_{(4)}, w_{(3)}) e^{-(\text{wt } \alpha_n(w'_{(4)}, w_{(3)}) + \text{wt } w'_{(4)} - \text{wt } w_{(3)}) \log z} \right)
\end{aligned} \tag{14.49}$$

when (14.7) holds and  $0 < |z| < \frac{|z_1 - z_2|}{2|z_2|}$ . We need the following lemma:

**Lemma 14.5** *Let  $f(z)$  be a multi-valued complex analytic function of  $z$  in a neighborhood of 0 with 0 deleted. If for  $z$  in this neighborhood such that  $\arg z$  satisfies  $0 \leq \arg z < 2\pi$ ,  $f(z)$  can be expanded as*

$$\sum_{i \in \mathbb{Z}_+} a_i e^{m_i \log z} \tag{14.50}$$

where  $\{m_i\}_{i \in \mathbb{Z}_+}$  is a sequence of strictly increasing real numbers, then the sequence  $\{m_i\}_{i \in \mathbb{Z}_+}$  and the coefficients  $a_i$ ,  $i \in \mathbb{Z}_+$ , are uniquely determined by  $f(z)$ .

The proof of this result will be given in the next section.

By assumption, we see that for  $z$  in this neighborhood such that  $\arg z$  satisfies  $0 \leq \arg z < 2\pi$ ,

$$\left( \sum_{n \in \mathbb{C}} \alpha_n(w'_{(4)}, w_{(3)}) e^{-(\text{wt } \alpha_n(w'_{(4)}, w_{(3)}) + \text{wt } w'_{(4)} - \text{wt } w_{(3)}) \log z} \right) (w_{(1)} \otimes w_{(2)})$$

for any  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ , as a multi-valued function of  $z$ , satisfies the condition in Lemma 14.5. Thus the coefficients of both sides of (14.49) in powers of  $x$  and  $x_1$  applied to  $w_{(1)} \otimes w_{(2)}$  also satisfies the conditions of Lemma 14.5. By Lemma 14.5, the coefficients of the left-hand side and of the right-hand side of (14.49) in powers of  $e^{\log z}$  are equal. Since we have already shown that for any  $n \in \mathbb{C}$ ,

$$x_1^{-1} \delta \left( \frac{x^{-1} - (z_1 - z_2)}{x_1} \right) Y'_{P(z_1 - z_2)}(v, x) \alpha_n(w'_{(4)}, w_{(3)})$$

exists, we obtain

$$\begin{aligned} \tau_{P(z_1-z_2)} \left( x_1^{-1} \delta \left( \frac{x^{-1} - (z_1 - z_2)}{x_1} \right) Y_t(v, x) \right) \alpha_n(w'_{(4)}, w_{(3)}) &= \\ = x_1^{-1} \delta \left( \frac{x^{-1} - (z_1 - z_2)}{x_1} \right) Y'_{P(z_1-z_2)}(v, x) \alpha_n(w'_{(4)}, w_{(3)}). \end{aligned} \quad (14.51)$$

proving that  $\alpha_n(w'_{(4)}, w_{(3)})$ ,  $n \in \mathbb{C}$ , satisfy the  $P(z_1 - z_2)$ -compatibility condition.

So  $\alpha_n(w'_{(4)}, w_{(3)})$ ,  $n \in \mathbb{C}$ , are in fact elements of  $W_1 \boxtimes_{P(z_1-z_2)} W_2$  and (14.42) can be regarded as an element of  $\overline{W_1 \boxtimes_{P(z_1-z_2)} W_2}$ . Let  $\mathcal{Y}_3$  be the intertwining operator corresponding to the  $P(z_1 - z_2)$ -intertwining map  $\boxtimes_{P(z_1-z_2)}$ . We have:

**Lemma 14.6** *Let  $G \in \text{Hom}(W_1 \boxtimes_{P(z_1-z_2)} W_2, (W'_{(4)} \otimes W_3)^*)$  be defined by*

$$(G(w))(w'_{(4)} \otimes w_{(3)}) = \langle \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}, w_{(3)})}^{(2)}, \nu \rangle_{W_1 \boxtimes_{P(z_1-z_2)} W_2} \quad (14.52)$$

for  $w \in W_1 \boxtimes_{P(z_1-z_2)} W_2$ ,  $w'_{(4)} \in W'_4$  and  $w_{(3)} \in W_3$ . Then  $G$  intertwines the two actions  $\tau_{W_1 \boxtimes_{P(z_1-z_2)} W_2}$  and  $\tau_{Q(z_2)}$  of  $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z+t)^{-1}]$  on  $W_1 \boxtimes_{P(z_1-z_2)} W_2$  and on  $(W'_{(4)} \otimes W_3)^*$ .

This result will be proved in the next section.

By this lemma and Remark 5.4 in [HL3] (Remark I.10 in [HL6]), there exists an intertwining operator  $\mathcal{Y}_4$  of type  $\binom{W_4}{W_1 \boxtimes_{P(z_1-z_2)} W_2 \ W_3}$  such that

$$(G(w))(w'_{(4)} \otimes w_{(3)}) = \langle w'_{(4)}, \mathcal{Y}_4(w, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_2=z_2} \quad (14.53)$$

for  $w \in W_1 \boxtimes_{P(z_1-z_2)} W_2$  and  $w'_{(4)} \in W'_4$  and  $w_{(3)} \in W_3$ . By Remark 13.2 in [HL6], (14.52), (14.53) and the absolute convergence of iterates of intertwining operators,

$$\begin{aligned} & \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}, w_{(3)})}^{(2)}(w_{(1)} \otimes w_{(2)}) \\ &= \langle \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}, w_{(3)})}^{(2)}, \mathcal{Y}_3(w_{(1)}, x_0) w_{(2)} \rangle_{W_1 \boxtimes_{P(z_1-z_2)} W_2} \Big|_{x_0=z_1-z_2} \\ &= (G(\mathcal{Y}_3(w_{(1)}, x_0) w_{(2)}))(w'_{(4)}, w_{(3)}) \Big|_{x_0=z_1-z_2} \\ &= \langle w'_{(4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)} x_0) w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_0=z_1-z_2, x_2=z_2} \end{aligned} \quad (14.54)$$

for any  $w'_{(4)} \in W_4$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$  and any  $z_1, z_2 \in \mathbb{C}$  satisfying (14.7). Let  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  be the intertwining operators corresponding to  $F_1$  and  $F_2$ , respectively. Then for any  $w'_{(4)} \in W_4$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$  and any  $z_1, z_2 \in \mathbb{C}$  satisfying (14.7), we obtain

$$\begin{aligned} & \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\ &= \langle w'_{(4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_0) w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_0=z_1-z_2, x_2=z_2}. \end{aligned} \quad (14.55)$$

Similarly, if in addition to the assumptions that  $V$  has one of the properties in Proposition 14.1, that  $V$  is rational and that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded, we assume that the vertex operator algebra  $V$  has the property that for any modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , any  $z_1, z_2 \in \mathbb{C}$  satisfying (14.7), any  $P(z_1 - z_2)$ - and  $P(z_2)$ -intertwining maps  $F_3$  and  $F_4$  of the types as above and any  $w'_{(4)} \in W'_4$ , the element  $\gamma(F_4; F_3, I)'(w'_{(4)})$  of  $(W_1 \otimes W_2 \otimes W_3)^*$  satisfies the  $P(z_2)$ -local grading-restriction condition, we can show that for any intertwining operators  $\mathcal{Y}_3$  and  $\mathcal{Y}_4$  of the same types as those for  $F_3$  and  $F_4$ , respectively, we can show that there exist intertwining operators  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  such that (14.55) holds for any  $w'_{(4)} \in W_4$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$  and any  $z_1, z_2 \in \mathbb{C}$  satisfying (14.7). .

We still assume, in addition to the assumptions that  $V$  has one of the properties in Proposition 14.1, that  $V$  is rational and that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded, that the vertex operator algebra  $V$  has the property that for any modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , any  $z_1, z_2 \in \mathbb{C}$  satisfying (14.7), any  $P(z_1)$ - and  $P(z_2)$ -intertwining maps  $F_1$  and  $F_2$  of the types as above and any  $w'_{(4)} \in W'_4$ , the element  $\gamma(F_1; I, F_2)'(w'_{(4)})$  in  $(W_1 \otimes W_2 \otimes W_3)^*$  satisfies the  $P(z_1 - z_2)$ -local grading restriction condition. For any  $P(z_1 - z_2)$ -intertwining map  $F_3$  and  $P(z_2)$ -intertwining map  $F_4$  of the types as above, when (14.7) holds, using (14.6) and  $\Omega_0$  and  $\Omega_{-1}$ , we have

$$\begin{aligned} & (\gamma(F_4; F_3, I)'(w'_{(4)}))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = \\ &= \langle e^{z_2 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}_4)(w_{(3)}, x_2) \cdot \\ & \quad \cdot \Omega_0(\Omega_{-1}(\mathcal{Y}_3))(w_{(1)}, x_0) w_{(2)} \rangle_{W_4} \Big|_{x_0=z_1-z_2, x_2=e^{\pi i} z_2} \\ &= \langle e^{z_2 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}_4)(w_{(3)}, x_2) e^{x_0 L(-1)} \cdot \\ & \quad \cdot \Omega_{-1}(\mathcal{Y}_3)(w_{(2)}, e^{\pi i} x_0) w_{(1)} \rangle_{W_4} \Big|_{x_0=z_1-z_2, x_2=e^{\pi i} z_2} \end{aligned}$$

$$\begin{aligned}
&= \langle e^{z_2 L'(1)} w'_{(4)}, e^{x_0 L(-1)} \Omega_{-1}(\mathcal{Y}_4)(w_{(3)}, x_2 - x_0) \cdot \\
&\quad \cdot \Omega_{-1}(\mathcal{Y}_3)(w_{(2)}, e^{\pi i} x_0) w_{(1)} \rangle_{W_4} \Big|_{x_0=z_1-z_2, x_2=e^{\pi i} z_2} \\
&= \langle e^{z_1 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}_4)(w_{(3)}, x_1) \cdot \\
&\quad \cdot \Omega_{-1}(\mathcal{Y}_3)(w_{(2)}, x_0) w_{(1)} \rangle_{W_4} \Big|_{x_1=e^{\pi i} z_1, x_0=e^{\pi i}(z_1-z_2)}. \quad (14.56)
\end{aligned}$$

By the proof of (14.55) above, there exist a module  $W_6$  and intertwining operators  $\mathcal{Y}_5$  and  $\mathcal{Y}_6$  of type  $\begin{pmatrix} W_6 \\ W_3 W_2 \end{pmatrix}$  and  $\begin{pmatrix} W_4 \\ W_6 W_1 \end{pmatrix}$ , respectively, such that when  $|z_1| > |z_1 - z_2| > |z_2| > 0$ ,

$$\begin{aligned}
&\langle e^{z_1 L'(1)} w'_{(4)}, \mathcal{Y}_6(\mathcal{Y}_5(w_{(3)}, x_2) w_{(2)}, x_0) w_{(1)} \rangle_{W_4} \Big|_{x_2=e^{\pi i} z_2, x_0=e^{\pi i}(z_1-z_2)} \\
&= \langle e^{z_1 L'(1)} w'_{(4)}, \Omega_{-1}(\mathcal{Y}_4)(w_{(3)}, x_1) \cdot \\
&\quad \cdot \Omega_{-1}(\mathcal{Y}_3)(w_{(2)}, x_0) w_{(1)} \rangle_{W_4} \Big|_{x_1=e^{\pi i} z_1, x_0=e^{\pi i}(z_1-z_2)}. \quad (14.57)
\end{aligned}$$

On the other hand, when  $|z_1| > |z_1 - z_2| > |z_2| > 0$ ,

$$\begin{aligned}
&\langle e^{z_1 L'(1)} w'_{(4)}, \mathcal{Y}_6(\mathcal{Y}_5(w_{(3)}, x_2) w_{(2)}, x_0) w_{(1)} \rangle_{W_4} \Big|_{x_2=e^{\pi i} z_2, x_0=e^{\pi i}(z_1-z_2)} \\
&= \langle e^{z_1 L'(1)} w'_{(4)}, \mathcal{Y}_6(\Omega_0(\Omega_{-1}(\mathcal{Y}_5))(w_{(3)}, x_2) \cdot \\
&\quad \cdot w_{(2)}, x_0) w_{(1)} \rangle_{W_4} \Big|_{x_2=e^{\pi i} z_2, x_0=e^{\pi i}(z_1-z_2)} \\
&= \langle e^{z_1 L'(1)} w'_{(4)}, \mathcal{Y}_6(e^{x_2 L(-1)} \Omega_{-1}(\mathcal{Y}_5)(w_{(2)}, e^{\pi i} x_2) \cdot \\
&\quad \cdot w_{(3)}, x_0) w_{(1)} \rangle_{W_4} \Big|_{x_2=e^{\pi i} z_2, x_0=e^{\pi i}(z_1-z_2)} \\
&= \langle e^{z_1 L'(1)} w'_{(4)}, \mathcal{Y}_6(\Omega_{-1}(\mathcal{Y}_5)(w_{(2)}, e^{\pi i} x_2) \cdot \\
&\quad \cdot w_{(3)}, x_0 + x_2) w_{(1)} \rangle_{W_4} \Big|_{x_2=e^{\pi i} z_2, x_0=e^{\pi i}(z_1-z_2)} \\
&= \langle e^{z_1 L'(1)} w'_{(4)}, \mathcal{Y}_6(\Omega_{-1}(\mathcal{Y}_5)(w_{(2)}, x_2) w_{(3)}, x_1) w_{(1)} \rangle_{W_4} \Big|_{x_1=e^{\pi i} z_1, x_2=e^{2\pi i} z_2}. \quad (14.58)
\end{aligned}$$

By the  $L(-1)$ -derivative property for intertwining operators, both sides of (14.56), (14.57) and (14.58) are analytic (multi-valued) functions of  $z_1$  and



$z_2$ . By these three formulas, we see that the left-hand side of (14.56) is equal to the value of an branch of the analytic extension of the right-hand side of (14.58) at  $(z_1, z_2)$  satisfying (14.7). Thus we can find  $p, q \in \mathbb{Z}$  such that if we let

$$\mathcal{Y}_7(\cdot, x) = \Omega_{-1}(\mathcal{Y}_5)(\cdot, e^{2\pi p i} x), \quad (14.59)$$

$$\mathcal{Y}_8(\cdot, x) = \mathcal{Y}_6(\cdot, e^{2\pi q i} x) \quad (14.60)$$

which are obviously also intertwining operators and let  $F_7$  and  $F_8$  the  $P(-z_1)$ - and  $P(z_2)$ -intertwining maps corresponding to  $\mathcal{Y}_7$  and  $\mathcal{Y}_8$ , respectively, then when (14.7) holds,

$$\begin{aligned} & (\gamma(F_4; F_3, I)'(w'_{(4)}))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = \\ & = \langle e^{z_1 L'(1)} w'_{(4)}, \mathcal{Y}_8(\mathcal{Y}_7(w_{(2)}, x_2) w_{(3)}, x_1) w_{(1)} \rangle_{W_4} \Big|_{x_1 = e^{\pi i} z_1, x_2 = z_2} \cdot \\ & = (\gamma(F_8; F_7, I)'(e^{z_1 L'(1)} w'_{(4)}))(w_{(2)} \otimes w_{(3)} \otimes w_{(1)}) \end{aligned} \quad (14.61)$$

Since  $(\gamma(F_8; F_7, I)'(e^{z_1 L'(1)} w'_{(4)}))$  satisfies the  $P(z_2)$ -local grading-restriction condition, by this formula, we see that  $\gamma(F_3; F_4, I)'(w'_{(4)})$  satisfies the  $P(z_2)$ -local grading-restriction condition. We have shown half of the following result:

**Proposition 14.7** *Assume that  $V$  is a rational vertex operator algebra, that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded and that  $V$  has one of the properties in Proposition 14.1. Then the following two properties are equivalent:*

1. *For any  $V$ -modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , any nonzero complex numbers  $z_1$  and  $z_2$  satisfying (14.7), any  $P(z_1)$ -intertwining map  $F_1$  of type  $\binom{W_4}{W_1 W_5}$  and  $P(z_2)$ -intertwining map  $F_2$  of type  $\binom{W_5}{W_2 W_3}$  and any  $w'_{(4)} \in W'_4$ ,  $\gamma(F_1; I, F_2)'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$  satisfies the  $P(z_1 - z_2)$ -local grading-restriction condition.*
2. *For any  $V$ -modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , any nonzero complex numbers  $z_1$  and  $z_2$  satisfying (14.7) and any  $P(z_1 - z_2)$ -intertwining map  $F_3$  of type  $\binom{W_4}{W_5 W_3}$  and  $P(z_2)$ -intertwining map  $F_4$  of type  $\binom{W_5}{W_1 W_2}$  and any  $w'_{(4)} \in W'_4$ ,  $\gamma(F_4; F_3, I)'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$  satisfies the  $P(z_2)$ -local grading-restriction condition.  $\square$*

The other half of this result can be proved similarly.

In fact much more has been proved. We have proved half of the following result on the associativity of intertwining operators:

**Theorem 14.8** *Assume that  $V$  is a rational vertex operator algebra, that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded and that  $V$  has one of the properties in Proposition 14.1. If  $V$  also has one of the properties in Proposition 14.7, then the intertwining operators for  $V$  have the following two associativity properties:*

1. *For any modules  $W_1, W_2, W_3, W_4$  and  $W_5$  and any intertwining operators  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  of type  $\binom{W_4}{W_1W_5}$  and  $\binom{W_5}{W_2W_3}$ , respectively, there exist a module  $W_6$  and intertwining operators  $\mathcal{Y}_3$  and  $\mathcal{Y}_4$  of type  $\binom{W_6}{W_1W_2}$  and  $\binom{W_4}{W_6W_3}$ , respectively, such that for any  $z_1, z_2 \in \mathbb{C}$  satisfying (14.7), (14.55) holds for any  $w'_{(4)} \in W'_4$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ .*
2. *For any modules  $W_1, W_2, W_3, W_4$  and  $W_6$  and any intertwining operators  $\mathcal{Y}_3$  and  $\mathcal{Y}_4$  of type  $\binom{W_6}{W_1W_2}$  and  $\binom{W_4}{W_6W_3}$ , respectively, there exist a module  $W_5$  and intertwining operators  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  of type  $\binom{W_4}{W_1W_5}$  and  $\binom{W_5}{W_2W_3}$ , respectively, such that for any  $z_1, z_2 \in \mathbb{C}$  satisfying (14.7), (14.55) holds for any  $w'_{(4)} \in W'_4$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ .*

*Conversely, If the intertwining operators for  $V$  have one of the associativity properties above, then  $V$  has both of the properties of Proposition 14.7, and in particular, the intertwining operators for  $V$  have both of the associativity properties above.*

*Proof* We need only to prove the second half of the result. Assume that the intertwining operators for  $V$  have the first associativity property in the theorem. For any  $V$ -modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , any nonzero complex numbers  $z_1$  and  $z_2$  satisfying (14.7), any  $P(z_1)$ -intertwining map  $F_1$  of type  $\binom{W_4}{W_1W_5}$  and  $P(z_2)$ -intertwining map  $F_2$  of type  $\binom{W_5}{W_2W_3}$  and any  $w'_{(4)} \in W'_4$ , by the first associativity property and the isomorphism between the space of intertwining operators and the space of intertwining maps, there exist a  $V$ -module  $W_6$  and  $P(z_1 - z_2)$ - and  $P(z_2)$ -intertwining maps  $F_3$  and  $F_4$  of type  $\binom{W_4}{W_6W_3}$  and  $\binom{W_6}{W_1W_2}$  such that

$$\gamma(F_1; I, F_2)' = \gamma(F_4; F_3, I)'. \quad (14.62)$$

We already know that for any  $w'_{(4)}$ ,  $\gamma(F_4; F_3, I)'(w'_{(4)})$  satisfies the  $P(z_1 - z_2)$ -local grading-restriction condition. So  $\gamma(F_1; I, F_2)'(w'_{(4)})$  also satisfies the  $P(z_1 - z_2)$ -local grading-restriction condition. Similarly we can prove the second half when the intertwining operators for  $V$  have the other associativity property.  $\square$

Now we assume that (14.7) holds and that  $V$  is rational, that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded and that  $V$  has either one of the properties in Proposition 14.1 and also either one of the properties in Proposition 14.7. Thus  $V$  satisfies all the four properties by these propositions.

We denote the graded space of all elements of  $(W_1 \otimes W_2 \otimes W_3)^*$  satisfying the  $P(z_1, z_2)$ -compatibility condition, the  $P(z_1, z_2)$ -, the  $P(z_2)$ - and the  $P(z_1 - z_2)$ -local grading-restriction conditions by  $W_{P(z_1, z_2)}$ .

The product of the  $P(z_1)$ -intertwining map  $\boxtimes_{P(z_1)}$  of type

$$\begin{pmatrix} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \\ W_1 \quad \quad \quad W_2 \boxtimes_{P(z_2)} W_3 \end{pmatrix}$$

and the  $P(z_2)$ -intertwining map  $\boxtimes_{P(z_2)}$  of the type  $\begin{pmatrix} W_2 \boxtimes_{P(z_2)} W_3 \\ W_2 \quad \quad \quad W_3 \end{pmatrix}$  gives a linear map  $\Psi_{P(z_1, z_2)}^{(1)}$  from

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) = (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

to  $(W_1 \otimes W_2 \otimes W_3)^*$  such that

$$\begin{aligned} & \Psi_{P(z_1, z_2)}^{(1)}(\nu)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\ &= \langle \nu, w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \end{aligned} \quad (14.63)$$

for all  $\nu \in W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ , where  $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$  is the image of  $w_{(1)} \otimes w_{(2)} \otimes w_{(3)}$  under the product of  $\boxtimes_{P(z_1)}$  and  $\boxtimes_{P(z_2)}$ . Since the images of elements of  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$  under  $\Psi_{P(z_1, z_2)}^{(1)}$  satisfy the the  $P(z_1, z_2)$ -compatibility condition and the  $P(z_1, z_2)$ -, the  $P(z_2)$ - and the  $P(z_1 - z_2)$ -local grading-restriction conditions,  $\Psi_{P(z_1, z_2)}^{(1)}$  is in fact a map to  $W_{P(z_1, z_2)}$ . From the definitions of  $L'_{P(z_1, z_2)}(0)$  and of  $Y'_{P(z_1, z_2)}$ , we see that  $\Psi_{P(z_1, z_2)}^{(1)}$  is also a graded linear map and we have

$$\Psi_{P(z_1, z_2)}^{(1)} Y'_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}(v, x) (\Psi_{P(z_1, z_2)}^{(1)})^{-1} = Y'_{P(z_1, z_2)}(v, x) \quad (14.64)$$

for all  $v \in V$ . Thus the image of  $\Psi_{P(z_1, z_2)}^{(1)}$  equipped with the vertex operator map  $Y'_{P(z_1, z_2)}$  is a  $V$ -module and  $\Psi_{P(z_1, z_2)}^{(1)}$  is a homomorphism of  $V$ -modules from  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$  to this  $V$ -module. We now want to show that the image of  $\Psi_{P(z_1, z_2)}^{(1)}$  is equal to  $W_{P(z_1, z_2)}$  and  $\Psi_{P(z_1, z_2)}^{(1)}$  is an isomorphism of  $V$ -modules from  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$  to  $W_{P(z_1, z_2)}$ .

We shall show this by constructing a linear map  $(\Psi_{P(z_1, z_2)}^{(1)})^{-1}$  from  $W_{P(z_1, z_2)}$  to  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$  and showing that it is the inverse of  $\Psi_{P(z_1, z_2)}^{(1)}$ . For any  $w_{(1)} \in W_1$  and any  $\lambda \in W_{P(z_1, z_2)}$ ,  $\mu_{\lambda, w_{(1)}}^{(1)}$  is in  $\overline{W_2 \boxtimes_{P(z_2)} W_3}$  by the  $P(z_1, z_2)$ -compatibility condition, the  $P(z_1, z_2)$ -local grading-restriction condition and the  $P(z_2)$ -local grading-restriction condition. We define an element  $(\Psi_{P(z_1, z_2)}^{(1)})^{-1}(\lambda) \in (W_1 \otimes (W_2 \boxtimes_{P(z_2)} W_3))^*$  by

$$(\Psi_{P(z_1, z_2)}^{(1)})^{-1}(\lambda)(w_{(1)} \otimes w) = \langle w, \mu_{\lambda, w_{(1)}}^{(1)} \rangle_{W_2 \boxtimes_{P(z_2)} W_3} \quad (14.65)$$

for all  $w_{(1)} \in W_1$  and  $w \in W_2 \boxtimes_{P(z_2)} W_3$ . By the  $P(z_1, z_2)$ -compatibility condition and the  $P(z_1, z_2)$ -local grading-restriction condition for  $\lambda$ ,  $(\Psi_{P(z_1, z_2)}^{(1)})^{-1}(\lambda)$  is in fact in  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ . Thus  $\lambda \mapsto (\Psi_{P(z_1, z_2)}^{(1)})^{-1}(\lambda)$  defines a map  $(\Psi_{P(z_1, z_2)}^{(1)})^{-1}$  from  $W_{P(z_1, z_2)}$  to  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ .

For any  $\lambda \in W_{P(z_1, z_2)}$ , using the definitions of  $\Psi_{P(z_1, z_2)}^{(1)}$ ,  $(\Psi_{P(z_1, z_2)}^{(1)})^{-1}$ ,  $\boxtimes_{P(z_1)}$  and  $\boxtimes_{P(z_2)}$ , we have

$$\begin{aligned} & \Psi_{P(z_1, z_2)}^{(1)}((\Psi_{P(z_1, z_2)}^{(1)})^{-1}(\lambda))(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\ &= \sum_{n \in \mathbb{C}} \langle (\Psi_{P(z_1, z_2)}^{(1)})^{-1}(\lambda), w_{(1)} \boxtimes_{P(z_1)} P_n(w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \\ &= \sum_{n \in \mathbb{C}} \boxtimes_{P(z_1)}((\Psi_{P(z_1, z_2)}^{(1)})^{-1}(\lambda))(w_{(1)} \otimes P_n(w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \\ &= \sum_{n \in \mathbb{C}} ((\Psi_{P(z_1, z_2)}^{(1)})^{-1}(\lambda))(w_{(1)} \otimes P_n(w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \\ &= \sum_{n \in \mathbb{C}} \langle P_n(w_{(2)} \boxtimes_{P(z_2)} w_{(3)}), \mu_{\lambda, w_{(1)}}^{(1)} \rangle_{W_2 \boxtimes_{P(z_2)} W_3} \\ &= \sum_{n \in \mathbb{C}} \langle P_n(\mu_{\lambda, w_{(1)}}^{(1)}), w_{(2)} \boxtimes_{P(z_2)} w_{(3)} \rangle_{W_2 \boxtimes_{P(z_2)} W_3} \\ &= \sum_{n \in \mathbb{C}} (\boxtimes_{P(z_2)}(P_n(\mu_{\lambda, w_{(1)}}^{(1)})))(w_{(2)} \otimes w_{(3)}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n \in \mathbb{C}} (P_n(\mu_{\lambda, w_{(1)}}^{(1)}))(w_{(2)} \otimes w_{(3)}) \\
&= \mu_{\lambda, w_{(1)}}^{(1)}(w_{(2)} \otimes w_{(3)}) \\
&= \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}),
\end{aligned} \tag{14.66}$$

proving

$$\Psi_{P(z_1, z_2)}^{(1)}(\Psi_{P(z_1, z_2)}^{(1)})^{-1} = 1. \tag{14.67}$$

Next we want to show that  $\Psi_{P(z_1, z_2)}^{(1)}$  is injective. Let  $\nu_1, \nu_2$  be two elements of  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$  such that

$$\Psi_{P(z_1, z_2)}^{(1)}(\nu_1) = \Psi_{P(z_1, z_2)}^{(1)}(\nu_2), \tag{14.68}$$

that is,

$$\begin{aligned}
&\langle \nu_1, w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \\
&= \langle \nu_2, w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}
\end{aligned} \tag{14.69}$$

for all  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ . Let  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  be the intertwining operators of type  $\begin{pmatrix} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \\ W_1 \quad W_2 \boxtimes_{P(z_2)} W_3 \end{pmatrix}$  and  $\begin{pmatrix} W_2 \boxtimes_{P(z_2)} W_3 \\ W_2 \quad W_3 \end{pmatrix}$  corresponding to the intertwining maps  $\boxtimes_{P(z_1)}$  and  $\boxtimes_{P(z_2)}$ , respectively. Then (14.69) becomes

$$\begin{aligned}
&\langle \nu_1, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \Big|_{x_1=z_1, x_2=z_2} \\
&= \langle \nu_2, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
&\quad \cdot \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \Big|_{x_1=z_1, x_2=z_2}.
\end{aligned} \tag{14.70}$$

By (14.39), both sides of (14.70), or equivalently of (14.69), can be expanded as series in  $e^{\log z_2}$  of the form

$$\sum_{i \in \mathbb{Z}_+} a_i(z_1) e^{m_i \log z_2} \tag{14.71}$$

where  $\{m_i\}_{i \in \mathbb{Z}_+}$  is a sequence with of strictly increasing real numbers. For a multi-valued function  $f(z)$  which can be expanded as  $\sum_{i \in \mathbb{Z}_+} a_i e^{m_i \log z}$  in a neighborhood of 0 with 0 deleted, we define  $\text{Res}_z f(z)$  to be  $a_{i_0}$  if there is a positive integer  $i_0$  such that  $m_{i_0} = -1$  and to be 0 if there is no such

$i_0$ . Lemma 14.5 implies that  $\text{Res}_z f(z)$  is well defined when  $\{m_l\}_{l \in \mathbb{Z}_+}$  is a sequence of strictly increasing real numbers.

Let  $f_l(z)$  and  $f_r(z)$  be the multi-valued functions of  $z$  obtained from the left-hand side and the right-hand side, respectively, of (14.70) by replacing  $z_2$  be  $z$ . From the assumption that the product of a  $P(z_1)$ -intertwining map and a  $P(z)$ -intertwining map is convergent when  $0 < |z| < |z_1|$ , we see that  $f_l(z)$  and  $f_r(z)$  are well-defined multi-valued functions of  $z$  when  $0 < |z| < |z_1|$ . By the  $L(-1)$ -derivative property for intertwining operators, we see that  $f_l(z)$  and  $f_r(z)$  are analytic when  $0 < |z| < |z_1|$ .

Now we can apply Lemma 14.5 to  $f_l(z)$  and  $f_r(z)$  since both can be expanded in the form of (14.71) with  $z_2$  replaced by  $z$ . By this lemma, the expansion coefficients  $\text{Res}_z e^{m \log z} f_l(z)$  and  $\text{Res}_z e^{m \log z} f_r(z)$ ,  $m \in \mathbb{C}$ , of  $f_l(z)$  and  $f_r(z)$  are determined uniquely by  $f_l(z)$  and  $f_r(z)$ , respectively. But  $f_l(z)$  and  $f_r(z)$  are equal. So their expansion coefficients are also equal. Writing down these expansion coefficients explicitly, we obtain

$$\begin{aligned} & \left\langle \nu_1, \mathcal{Y}_1(w_{(1)}, x_1)(w_{(2)})_m w_{(3)} \right\rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \Big|_{x_1^n = e^{n \log z_1}, n \in \mathbb{C}} \\ &= \left\langle \nu_2, \mathcal{Y}_1(w_{(1)}, x_1)(w_{(2)})_m w_{(3)} \right\rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \Big|_{x_1^n = e^{n \log z_1}, n \in \mathbb{C}} \end{aligned} \quad (14.72)$$

or equivalently

$$\begin{aligned} & \left\langle \nu_1, w_{(1)} \boxtimes_{P(z_1)} (w_{(2)})_m w_{(3)} \right\rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \\ &= \left\langle \nu_2, w_{(1)} \boxtimes_{P(z_1)} (w_{(2)})_m w_{(3)} \right\rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \end{aligned} \quad (14.73)$$

for all  $m \in \mathbb{C}$  and all  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ . The conclusion  $\nu_1 = \nu_2$  which we need follows from the result below:

**Lemma 14.9** *The module  $W_2 \boxtimes_{P(z_2)} W_3$  is spanned by the homogeneous components of the elements of  $\overline{W_2 \boxtimes_{P(z_2)} W_3}$  of the form  $w_{(2)} \boxtimes_{P(z_2)} w_{(3)}$ , for all  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ .*

We shall prove this in the next section.

We have shown that  $\Psi_{P(z_1, z_2)}^{(1)}$  is injective. Combining this result with (14.53), we see that  $\Psi_{P(z_1, z_2)}^{(1)}$  is an isomorphism of  $V$ -modules from

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

to  $W_{P(z_1, z_2)}$ .

Similarly, the iterate of the  $P(z_1 - z_2)$ -intertwining map  $\boxtimes_{P(z_1 - z_2)}$  of type

$$\begin{pmatrix} (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3 \\ W_1 \boxtimes_{P(z_1 - z_2)} W_2 & W_3 \end{pmatrix}$$

and the  $P(z_4)$ -intertwining map  $\boxtimes_{P(z_4)}$  of type  $\begin{pmatrix} W_1 \boxtimes_{P(z_1 - z_2)} W_2 \\ W_1 & W_2 \end{pmatrix}$  gives a linear map  $\Psi_{P(z_1, z_2)}^{(2)}$  from

$$(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3 = ((W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3)'$$

to  $(W_1 \otimes W_2 \otimes W_3)^*$  such that

$$\begin{aligned} \Psi_{P(z_1, z_2)}^{(2)}(\nu)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) &= \\ &= \langle \nu, (w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \rangle_{(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3} \end{aligned} \quad (14.74)$$

for all

$$\nu \in (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3,$$

$w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ , where  $(w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$  is the image of  $w_{(1)} \otimes w_{(2)} \otimes w_{(3)}$  under the iterate of  $\boxtimes_{P(z_1 - z_2)}$  and  $\boxtimes_{P(z_2)}$ . We can show similarly that this map is in fact an isomorphism of  $V$ -modules from

$$(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

to  $W_{P(z_1, z_2)}$ .

Let

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1 - z_2), P(z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \rightarrow (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

be the graded adjoint map of  $(\Psi_{P(z_1, z_2)}^{(1)})^{-1} \circ \Psi_{P(z_1, z_2)}^{(2)}$ . Then  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1 - z_2), P(z_2)}$  is an isomorphism and is called the *associativity isomorphism for  $W_1$ ,  $W_2$  and  $W_3$  associated with  $(P(z_1), P(z_2); P(z_1 - z_2), P(z_2))$* . From (14.63) and (14.74), we obtain

$$\begin{aligned} \langle \nu, \overline{\mathcal{A}}_{P(z_1), P(z_2)}^{P(z_1 - z_2), P(z_2)}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \rangle_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \\ = \langle \nu, (w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \rangle_{(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3} \end{aligned} \quad (14.75)$$

for all  $\nu \in (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ , where

$$\overline{\mathcal{A}}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \rightarrow \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}$$

is the unique extension of  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$ . Equivalently, we obtain

$$\begin{aligned} & \overline{\mathcal{A}}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \\ &= (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \end{aligned} \quad (14.76)$$

for all  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ .

We summarize what we obtained above in half of the following theorem:

**Theorem 14.10** *Assume that  $V$  is a rational vertex operator algebra, that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded and that  $V$  has one of the properties in Proposition 14.1. If  $V$  also has either one of the properties in Proposition 14.7, then for any  $V$ -module  $W_1$ ,  $W_2$  and  $W_3$  and any complex numbers  $z_1$  and  $z_2$  satisfying (14.7), there exists a unique isomorphism  $\overline{\mathcal{A}}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$  from  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$  to  $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$  such that (14.76) holds for all  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ . Conversely, if there exists such an isomorphism for any  $V$ -module  $W_1$ ,  $W_2$  and  $W_3$  and any complex numbers  $z_1$  and  $z_2$  satisfying (14.7), then  $V$  has both properties in Proposition 14.7.*

*Proof* We need only to show that if there exists such an isomorphism for any  $V$ -module  $W_1$ ,  $W_2$  and  $W_3$  and any complex numbers  $z_1$  and  $z_2$  satisfying (14.7),  $V$  has the first property in Proposition 14.7.

Let  $W_{P(z_1), P(z_2)}$  be the subspace of  $(W_1 \otimes W_2 \otimes W_3)^*$  consisting of elements satisfying the  $P(z_1, z_2)$ -compatibility condition, the  $P(z_1, z_2)$ -local grading-restriction condition and the  $P(z_2)$ -local grading-restriction condition and  $W^{P(z_1-z_2), P(z_2)}$  the subspace of  $(W_1 \otimes W_2 \otimes W_3)^*$  consisting of elements satisfying the  $P(z_1, z_2)$ -compatibility condition, the  $P(z_1, z_2)$ -local grading-restriction condition and the  $P(z_1-z_2)$ -local grading-restriction condition. Since products and iterates of intertwining maps give elements in  $W_{P(z_1), P(z_2)}$  and  $W^{P(z_1-z_2), P(z_2)}$ , respectively, we need only to prove

$$W_{P(z_1), P(z_2)} = W^{P(z_1-z_2), P(z_2)} (= W_{P(z_1, z_2)}). \quad (14.77)$$



The maps  $\Psi_{P(z_1, z_2)}^{(1)}$  and  $(\Psi_{P(z_1, z_2)}^{(1)})^{-1}$  in this case are still well defined and are maps from

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

to  $W_{P(z_1), P(z_2)}$  and from  $W_{P(z_1), P(z_2)}$  to

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3),$$

respectively, and they are still the inverses of each other. Similarly we have the maps  $\Psi_{P(z_1, z_2)}^{(2)}$  and  $(\Psi_{P(z_1, z_2)}^{(2)})^{-1}$  from

$$(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

to  $W^{P(z_1-z_2), P(z_2)}$  and from  $W^{P(z_1-z_2), P(z_2)}$  to

$$(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3,$$

respectively, and they are the inverses of each other. Let

$$(\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)})' : (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \rightarrow W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

be the graded adjoint of  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$ . Then

$$\Psi_{P(z_1, z_2)}^{(1)} \circ (\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)})' \circ (\Psi_{P(z_1, z_2)}^{(2)})^{-1}$$

is an isomorphism from  $W^{P(z_1-z_2), P(z_2)}$  to  $W_{P(z_1), P(z_2)}$ . But on the other hand, by the definitions of  $(\Psi_{P(z_1, z_2)}^{(2)})^{-1}$  and  $\Psi_{P(z_1, z_2)}^{(1)}$  and (14.75), for any  $\lambda \in W^{P(z_1-z_2), P(z_2)}$ ,

$$\Psi_{P(z_1, z_2)}^{(1)} \circ (\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)})' \circ (\Psi_{P(z_1, z_2)}^{(2)})^{-1}(\lambda)$$

as an element of  $(W_1 \otimes W_2 \otimes W_3)^*$  is equal to  $\lambda$  itself, Thus we have (14.77).  $\square$

By Proposition 14.8 and this theorem, we obtain:

**Theorem 14.11** *Assume that  $V$  is a rational vertex operator algebra, that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded and that  $V$  has one of the properties in Proposition 14.1. Then for any  $V$ -module  $W_1$ ,  $W_2$  and  $W_3$  and any complex numbers  $z_1$  and  $z_2$  satisfying (14.7), there exists a unique isomorphism  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$  from  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$  to  $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$  such that (14.76) holds for all  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$  if and only if the intertwining operators for  $V$  have the associativity properties in Proposition 14.8.*

## 15 Proofs of the lemmas in Section 14

### 15.1 Proof of Lemma 14.2

We prove only (14.9). The Proofs of (14.8), (14.10) and (14.11) are similar. In the process of the proof, we shall see that both sides of (14.9) exist and their coefficients are in fact expansions of rational functions as indicated.

By the definition of the  $\delta$ -function, the right-hand side of (14.9) is equal to

$$\begin{aligned}
 & \sum_{n,m \in \mathbb{Z}} z_2^{-n-1} (x_0^{-1} - x_2)^n (z_1 - z_2)^{-m-1} (x_2 - x_1)^m \\
 &= \sum_{n,m \in \mathbb{Z}} \sum_{k,l \in \mathbb{N}} \binom{n}{k} \binom{m}{l} z_2^{-n-1} (z_1 - z_2)^{-m-1} x_0^{-n+k} (-1)^k x_2^k x_2^{m-l} (-1)^l x_1^l \\
 &= \sum_{n,m \in \mathbb{Z}} \sum_{k,l \in \mathbb{N}} \binom{n}{k} \binom{m}{l} (-1)^{k+l} z_2^{-n-1} (z_1 - z_2)^{-m-1} x_0^{-n+k} x_1^l x_2^{k+m-l}.
 \end{aligned} \tag{15.1}$$

The coefficient of the right-hand side of (15.1) in  $x_0^r x_1^s x_2^t$  is

$$\begin{aligned}
 & \sum_{k \in \mathbb{N}} \binom{k-r}{k} \binom{t-k+s}{s} (-1)^{k+s} z_2^{-k+r-1} (z_1 - z_2)^{-t+k-s-1} \\
 &= \sum_{k \in \mathbb{N}} \binom{r-1}{k} \binom{k-t-1}{s} z_2^{-k+r-1} (z_1 - z_2)^{-t+k-s-1} \\
 &= \sum_{k \in \mathbb{N}} \binom{r-1}{k} z_2^{-k+r-1} \frac{1}{s!} \frac{d^s}{dz_1^s} (z_1 - z_2)^{-t+k-1} \\
 &= \frac{1}{s!} \frac{d^s}{dz_1^s} \left( (z_1 - z_2)^{-t-1} \sum_{k \in \mathbb{N}} \binom{r-1}{k} z_2^{-k+r-1} (z_1 - z_2)^k \right). \tag{15.2}
 \end{aligned}$$

Since we have the inequality  $|z_2| > |z_1 - z_2|$ , the right-hand side of (15.2) is absolutely convergent to

$$\frac{1}{s!} \frac{d^s}{dz_1^s} \left( (z_1 - z_2)^{-t-1} z_1^{r-1} \right). \tag{15.3}$$

On the other hand, the left-hand side of (14.9) is equal to

$$\sum_{n,m \in \mathbb{Z}} z_1^{-n-1} (x_0^{-1} - x_1)^n x_2^{-m-1} (x_0^{-1} - z_2)^m$$

$$\begin{aligned}
&= \sum_{n,m \in \mathbb{Z}} \sum_{k,l \in \mathbb{N}} \binom{n}{k} \binom{m}{l} z_1^{-n-1} x_0^{-n+k} (-1)^k x_1^k x_2^{-m-1} x_0^{-m+l} (-1)^l z_2^l \\
&= \sum_{n,m \in \mathbb{Z}} \sum_{k,l \in \mathbb{N}} \binom{n}{k} \binom{m}{l} (-1)^{k+l} z_1^{-n-1} z_2^l x_0^{-n+k-m+l} x_1^k x_2^{-m-1}. \quad (15.4)
\end{aligned}$$

The coefficient of the right-hand side of (15.4) in  $x_0^r x_1^s x_2^t$  is

$$\begin{aligned}
&\sum_{l \in \mathbb{N}} \binom{-r+s+t+1+l}{s} \binom{-t-1}{l} (-1)^{s+l} z_1^{r-s-t-l-2} z_2^l \\
&= \sum_{l \in \mathbb{N}} \frac{1}{s!} \frac{d^s}{dz_1^s} \binom{-t-1}{l} (-1)^l z_1^{r-t-l-2} z_2^l \\
&= \frac{1}{s!} \frac{d^s}{dz_1^s} \left( z_1^{r-1} \sum_{l \in \mathbb{N}} \binom{-t-1}{l} (-1)^l z_1^{-t-1-l} z_2^l \right). \quad (15.5)
\end{aligned}$$

Since we also have the inequality  $|z_1| > |z_2|$ , the right-hand side of (15.5) is absolutely convergent to (15.3), proving (14.9).

## 15.2 Proof of Lemma 14.3

We prove only (14.36). The definition (14.18) can be written as

$$\begin{aligned}
&z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) (z_1 - z_2)^{-1} \delta \left( \frac{x_2 - x_1}{z_1 - z_2} \right) \cdot \\
&\quad \cdot \lambda(Y_1(e^{x_0 L(1)}(-x_0^2)^{-L(0)} v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
&+ z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) x_1^{-1} \delta \left( \frac{z_1 - z_2 - x_2}{x_1} \right) \cdot \\
&\quad \cdot \lambda(w_{(1)} \otimes Y_2(e^{x_0 L(1)}(-x_0^2)^{-L(0)} v, x_2) w_{(2)} \otimes w_{(3)}) \\
&= \left( \tau_{P(z_1, z_2)}^{(2)} \left( x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \cdot \right. \right. \\
&\quad \cdot x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) Y_t(v, x_0) \lambda \Big) (w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
&\quad \left. + x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) \cdot \right. \\
&\quad \left. \cdot \lambda(w_{(1)} \otimes w_{(2)} \otimes Y_3(e^{x_0 L(1)}(-x_0^2)^{-L(0)} v, x_0^{-1}) w_{(3)}) \right) \quad (15.6)
\end{aligned}$$

On the both sides of (15.6), first replacing  $v$  by

$$(-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v$$

and then taking  $\text{Res}_{x_0^{-1}}$ , we obtain

$$\begin{aligned} & (z_1 - z_2)^{-1} \delta \left( \frac{x_2 - x_1}{z_1 - z_2} \right) \cdot \\ & \quad \cdot \lambda(Y_1(e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\ & + x_1^{-1} \delta \left( \frac{z_1 - z_2 - x_2}{x_1} \right) \cdot \\ & \quad \cdot \lambda(w_{(1)} \otimes Y_2(e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_2) w_{(2)} \otimes w_{(3)}) \\ & = \text{Res}_{x_0^{-1}} \left( \tau_{P(z_1, z_2)}^{(2)} \left( x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) \right) \cdot \right. \\ & \quad \cdot Y_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0) \Big) \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\ & + \text{Res}_{x_0^{-1}} x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) \cdot \\ & \quad \cdot \lambda(w_{(1)} \otimes w_{(2)} \otimes Y_3(e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0^{-1}) w_{(3)}). \end{aligned} \quad (15.7)$$

By the definition of  $\tau_{P(z_1 - z_2)}$  and (15.7),

$$\begin{aligned} & \left( \tau_{P(z_1 - z_2)} \left( x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) Y_t(v, x_2^{-1}) \right) \mu_{\lambda, w_{(3)}}^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \\ & = \text{Res}_{x_0^{-1}} \left( \tau_{P(z_1, z_2)}^{(2)} \left( x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) \right) \cdot \right. \\ & \quad \cdot Y_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0) \Big) \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\ & + \text{Res}_{x_0^{-1}} x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{x_2} \right) x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) \cdot \\ & \quad \cdot \lambda(w_{(1)} \otimes w_{(2)} \otimes Y_3(e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0^{-1}) w_{(3)}). \end{aligned} \quad (15.8)$$

Note that the right-hand side of (15.8) exists. Since  $\lambda$  satisfies the  $P(z_1, z_2)$ -compatibility condition, the right-hand side of (15.8) is equal to

$$x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) \text{Res}_{x_0^{-1}} \left( \tau_{P(z_1, z_2)}^{(2)} \left( x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \right) \cdot \right.$$

$$\begin{aligned}
& \cdot Y_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0) \Big) \lambda \Big) (w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& + x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) \text{Res}_{x_0^{-1}} x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{x_2} \right) \cdot \\
& \cdot \lambda(w_{(1)} \otimes w_{(2)} \otimes Y_3(e^{x_2^{-1} L(1)} (-x_2^2)^{L(0)} v, x_0^{-1}) w_{(3)}) \quad (15.9)
\end{aligned}$$

Taking  $\text{Res}_{x_1}$  in (15.9) and then multiplying it by  $x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right)$ , we obtain (15.9) itself. But by the calculations (15.8)–(15.9), (15.9) is equal to the left-hand side of (15.8). So we see that the left-hand side of (15.8) is equal to

$$\begin{aligned}
& x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) (\tau_{P(z_1 - z_2)}(Y_t(v, x_2^{-1}) \mu_{\lambda, w_{(3)}}^{(2)})(w_{(1)} \otimes w_{(2)})) \\
& = x_1^{-1} \delta \left( \frac{x_2 - (z_1 - z_2)}{x_1} \right) (Y_{P(z_1 - z_2)}(v, x_2^{-1}) \mu_{\lambda, w_{(3)}}^{(2)})(w_{(1)} \otimes w_{(2)}), \quad (15.10)
\end{aligned}$$

proving (14.36).

### 15.3 Proof of Lemma 14.4

Since the vertex operator algebra is rational, the modules  $W_1$ ,  $W_2$  and  $W_3$  can be decomposed as direct sums of finitely many irreducible modules. Let

$$W_1 = \coprod_{i=1}^r M_i^{(1)}, \quad (15.11)$$

$$W_2 = \coprod_{i=1}^s M_i^{(2)}, \quad (15.12)$$

$$W_3 = \coprod_{i=1}^t M_i^{(3)} \quad (15.13)$$

where  $M_1^{(1)}, \dots, M_r^{(1)}$ ,  $M_1^{(2)}, \dots, M_s^{(2)}$  and  $M_1^{(3)}, \dots, M_t^{(3)}$  are irreducible  $V$ -modules. Let  $\eta_i$ ,  $\eta_j$  and  $\eta_k$  be the projections from  $W_2$  to  $M_i$ , from  $W_3$  to  $M_j$  and from  $W_2 \boxtimes W_3$  to  $M_k$ , respectively, for any positive integers  $i$ ,  $j$  and  $k$  less than or equal to  $r$ ,  $s$  and  $t$ , respectively. Then for such  $i$ ,  $j$  and  $k$ , it is clear that  $\eta_k \circ \mathcal{Y} \circ (\eta_i \otimes \eta_j)$  is an intertwining operator of type  $\begin{pmatrix} M_k \\ M_i M_j \end{pmatrix}$ .

Moreover, we have

$$\mathcal{Y} = \sum_{i=1}^r \sum_{j=1}^s \sum_{k=1}^t \eta_k \circ \mathcal{Y} \circ (\eta_i \otimes \eta_j). \quad (15.14)$$

Since  $\eta_k \circ \mathcal{Y} \circ (\eta_i \otimes \eta_j)$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $1 \leq k \leq t$ , are intertwining operators among irreducible modules, there exist complex numbers  $h_{ijk}$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $1 \leq k \leq t$ , such that  $\eta_k \circ \mathcal{Y} \circ (\eta_i \otimes \eta_j)$  are in fact maps from  $M_i \otimes M_j$  to  $x^{h_{ijk}} M_k[[x, x^{-1}]]$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $1 \leq k \leq t$  (see [FHL]). Also if all irreducible  $V$ -modules are  $\mathbb{R}$ -graded,  $h_{ijk}$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq s$ ,  $1 \leq k \leq t$ , are in fact real numbers. These facts together with (15.14) proves the first part of Lemma 14.4.

## 15.4 Proof of Lemma 14.5

We have

$$\begin{aligned} e^{-m_1(\log |z| + i \arg z)} f(z) &= \sum_{l \in \mathbb{Z}_+} a_l e^{(m_l - m_1)(\log |z| + i \arg z)} \\ &= a_1 + e^{(m_2 - m_1)(\log |z| + i \arg z)} g(z) \end{aligned} \quad (15.15)$$

where

$$g(z) = \sum_{l \in \mathbb{Z}_+ \setminus \{1\}} a_l e^{(m_l - m_2)(\log |z| + i \arg z)}. \quad (15.16)$$

Since  $\sum_{l \in \mathbb{Z}_+} a_l e^{m_l(\log |z| + i \arg z)}$  converges absolutely to  $f(z)$ , the right-hand side of (15.16) converges absolutely to a multi-valued function  $g(z)$ . Since  $m_l$ ,  $l \in \mathbb{Z}_+$ , are real, the absolute convergence of  $\sum_{l \in \mathbb{Z}_+} a_l e^{m_l(\log |z| + i \arg z)}$  when  $|z|$  is small means that  $\sum_{l \in \mathbb{Z}_+} |a_l| e^{m_l(\log |z|)}$  is convergent when  $|z|$  is small. Thus

$$\sum_{l \in \mathbb{Z}_+} |a_l| e^{(m_l - m_2)(\log |z|)} \quad (15.17)$$

is also convergent when  $|z|$  is small. Since  $m_l \geq m_2$ ,  $l \in \mathbb{Z}_+ \setminus \{1\}$ , we see that the limit of (15.17) when  $|z| \rightarrow 0$  exists. In particular, (15.17) is bounded. Since

$$\begin{aligned} |g(z)| &= \left| \sum_{l \in \mathbb{Z}_+ \setminus \{1\}} a_l e^{(m_l - m_2)(\log |z| + i \arg z)} \right| \\ &\leq \sum_{l \in \mathbb{Z}_+ \setminus \{1\}} |a_l| e^{(m_l - m_2)(\log |z|)}, \end{aligned} \quad (15.18)$$

we see that  $g(z)$  is also bounded. Taking  $\lim_{|z| \rightarrow 0}$  on both sides of (15.15), we obtain  $a_1 = \lim_{|z| \rightarrow 0} e^{-m_1(\log |z| + i \arg z)} f(z)$ . So  $a_1$  is uniquely determined by  $f(z)$  and  $m_1$ . Using induction, we can show that for any  $l \in \mathbb{Z}_+$ ,  $a_l$  is uniquely determined by  $f(z)$  and  $m_1, \dots, m_l$ .

We now show that  $\{m_l\}_{l \in \mathbb{Z}_+}$  is uniquely determined by  $f(z)$ . Assume that there is another sequence  $\{n_l\}_{l \in \mathbb{Z}_+}$  of strictly increasing real numbers such that  $f(z)$  can be expanded as  $\sum_{l \in \mathbb{Z}_+} b_l e^{n_l(\log |z| + i \arg z)}$ . If  $\{m_l\}_{l \in \mathbb{Z}_+}$  and  $\{n_l\}_{l \in \mathbb{Z}_+}$  are different, we can find  $l_0$  such that  $m_{l_0} \neq n_{l_0}$  but  $m_l = n_l$ ,  $l < l_0$ . Let

$$\tilde{f}(z) = \sum_{l \in \mathbb{Z}_+ \setminus \{1, \dots, l_0-1\}} b_l e^{n_l(\log |z| + i \arg z)}. \quad (15.19)$$

Since  $f(z)$  is also equal to  $\sum_{l \in \mathbb{Z}_+} a_l e^{m_l(\log |z| + i \arg z)}$  and  $m_l = n_l$ ,  $l < l_0$ , we have

$$\tilde{f}(z) = \sum_{l \in \mathbb{Z}_+ \setminus \{1, \dots, l_0-1\}} a_l e^{m_l(\log |z| + i \arg z)}. \quad (15.20)$$

Assume that  $m_{l_0} < n_{l_0}$ . By the discussion above we see that

$$b_{l_0} = \lim_{|z| \rightarrow 0} e^{-n_{l_0}(\log |z| + i \arg z)} \tilde{f}(z). \quad (15.21)$$

But by (15.20) and the assumption  $m_{l_0} < n_{l_0}$ , we see that the right-hand side of (15.21) does not exist. Contradiction. So  $\{m_l\}_{l \in \mathbb{Z}_+}$  and  $\{n_l\}_{l \in \mathbb{Z}_+}$  are equal.

## 15.5 Proof of Lemma 14.6

For any  $w \in W_1 \boxtimes_{P(z_1-z_2)} W_2$ ,

$$\langle w, \mu_{\gamma(F_1; I, F_2)(\cdot), \cdot}^{(2)} \rangle_{W_1 \boxtimes_{P(z_1-z_2)} W_2}$$

is an element of  $(W'_{(4)} \otimes W_3)^*$ . We want to show

$$\begin{aligned} & \langle z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) Y_{P(z_1-z_2)}(v, x) w, \mu_{\gamma(F_1; I, F_2)(\cdot), \cdot}^{(2)} \rangle_{W_1 \boxtimes_{P(z_1-z_2)} W_2} \\ &= \tau_{Q(z_2)} \left( z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) Y_t(v, x) \right) (\langle w, \mu_{\gamma(F_1; I, F_2)(\cdot), \cdot}^{(2)} \rangle_{W_1 \boxtimes_{P(z_1-z_2)} W_2}). \end{aligned} \quad (15.22)$$

By (15.6) and (15.7) with  $\lambda = \gamma(F_1; I, F_2)(w'_{(4)})$ , we have

$$\begin{aligned}
& \left( \tau_{P(z_1, z_2)}^{(2)} \left( x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \right. \right. \\
& \quad \cdot Y_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} v, x_0) \left. \left. \right) (\gamma(F_1; I, F_2)'(w'_{(4)})) \right) (w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& \quad + x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{x_2} \right) \cdot \\
& \quad \cdot (\gamma(F_1; I, F_2)'(w'_{(4)})) (w_{(1)} \otimes w_{(2)} \otimes Y(v, x_0^{-1}) w_{(3)}) \\
& = z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) \text{Res}_{x_0^{-1}} \left( \tau_{P(z_1, z_2)}^{(2)} \left( x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \right. \right. \\
& \quad \cdot Y'_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} v, x_0) \left. \left. \right) (\gamma(F_1; I, F_2)'(w'_{(4)})) \right) (w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& \quad + z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) \text{Res}_{x_0^{-1}} x_2^{-1} \delta \left( z_2 - \frac{x_0^{-1}}{x_2} \right) \cdot \\
& \quad \cdot (\gamma(F_1; I, F_2)'(w'_{(4)})) (w_{(1)} \otimes w_{(2)} \otimes Y(v, x_0^{-1}) w_{(3)}). \tag{15.23}
\end{aligned}$$

By (14.14), the definition of  $\gamma(F_1; I, F_2)'$  and the definitions of  $\tau_{P(z_1, z_2)}$  and  $\mu_{\cdot, \cdot}^{(2)}$ , the left-hand side of (15.23) is equal to

$$\begin{aligned}
& x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \cdot \\
& \quad \cdot (\gamma(F_1; I, F_2)'(Y'_4((-x_0^2)^{L(0)} e^{-x_0 L(1)} v, x_0) w'_{(4)})) (w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& \quad + x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{x_2} \right) \cdot \\
& \quad \cdot (\gamma(F_1; I, F_2)'(w'_{(4)})) (w_{(1)} \otimes w_{(2)} \otimes Y(v, x_0^{-1}) w_{(3)}) \\
& = x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \mu_{\gamma(F_1; I, F_2)'(Y'_4(v, x_0^{-1}) w'_{(4)}), w_{(3)}}^{(2)} (w_{(1)} \otimes w_{(2)}) \\
& \quad + x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{x_2} \right) \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}), Y(v, x_0^{-1}) w_{(3)}}^{(2)} (w_{(1)} \otimes w_{(2)}). \tag{15.24}
\end{aligned}$$

On the other hand, taking  $\text{Res}_{x_1}$  in (15.8) with  $\lambda = \gamma(F_1; I, F_2)(w'_{(4)})$ , replacing  $v$  in the result by  $(-x_2^{-2})^{L(0)} e^{-x_2^{-1} L(1)} v$  and then multiplying the result



by  $z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right)$ , we obtain

$$\begin{aligned}
& \left( z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) Y'_{P(z_1 - z_2)}((-x_2^{-2})^{L(0)} e^{-x_2^{-1} L(1)} v, x_2^{-1}) \right. \\
& \quad \left. \cdot \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}), w_{(3)}}^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \\
&= z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) \text{Res}_{x_0^{-1}} (\tau_{P(z_1, z_2)}^{(2)} (x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \\
& \quad \cdot Y'_t((-x_0^2)^{L(0)} e^{-x_0 L(1)} v, x_0)) (\gamma(F_1; I, F_2)'(w'_{(4)}))) (w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
& \quad + z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) \text{Res}_{x_0^{-1}} x_2^{-1} \delta \left( z_2 - \frac{x_0^{-1}}{x_2} \right) \\
& \quad \cdot (\gamma(F_1; I, F_2)'(w'_{(4)})) (w_{(1)} \otimes w_{(2)} \otimes Y(v, x_0^{-1}) w_{(3)}). \tag{15.25}
\end{aligned}$$

From the calculations (15.23)–(15.25), we obtain

$$\begin{aligned}
& \left( z_2^{-1} \delta \left( \frac{x_0^{-1} - x_2}{z_2} \right) Y'_{P(z_1 - z_2)}((-x_2^{-2})^{L(0)} e^{-x_2^{-1} L(1)} v, x_2^{-1}) \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}), w_{(3)}}^{(2)} \right) \\
&= x_2^{-1} \delta \left( \frac{x_0^{-1} - z_2}{x_2} \right) \mu_{\gamma(F_1; I, F_2)'(Y_4^*(v, x_0^{-1}) w'_{(4)}), w_{(3)}}^{(2)} \\
& \quad + x_2^{-1} \delta \left( \frac{z_2 - x_0^{-1}}{x_2} \right) \mu_{\gamma(F_1; I, F_2)'(w'_{(4)}), Y(v, x_0^{-1}) w_{(3)}}^{(2)}. \tag{15.26}
\end{aligned}$$

This formula is equivalent to (15.22).

## 15.6 Proof of Lemma 14.9

Assume that the module spanned by the homogeneous components of the elements of  $W_2 \boxtimes_{P(z_2)} W_3$  of the form  $w_{(2)} \boxtimes_{P(z_2)} w_{(3)}$  for all  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$  is  $W_0$ . We want to show that  $W_0 = W_2 \boxtimes_{P(z_2)} W_3$ .

If not, the quotient module

$$W = (W_2 \boxtimes_{P(z_2)} W_3) / W_0 \tag{15.27}$$

is nontrivial. Let  $P_W$  be the projection from  $W_2 \boxtimes_{P(z_2)} W_3$  to  $W$ . Since  $W$  is nontrivial,  $P_W$  is a nontrivial module map. By Proposition 12.3 in [HL6],  $\eta \rightarrow \bar{\eta} \circ \boxtimes_{P(z_2)}$  is a linear isomorphism from the space of module maps from

$W_2 \boxtimes_{P(z_2)} W_3$  to  $W$  to the space of  $P(z_2)$ -intertwining maps of type  $\begin{pmatrix} W \\ W_2 W_3 \end{pmatrix}$ . Thus  $\overline{P}_W \circ \boxtimes_{P(z_2)}$  is a nontrivial  $P(z_2)$ -intertwining maps of type  $\begin{pmatrix} W \\ W_2 W_3 \end{pmatrix}$ . But since the image of  $\boxtimes_{P(z_2)}$  is in  $\overline{W}_0$ ,  $\overline{P}_W \circ \boxtimes_{P(z_2)}$  is the trivial map. We have a contradiction.

## 16 Sufficient conditions for the existence of associativity isomorphisms

In Section 14, we have constructed the associativity isomorphisms for  $P(\cdot)$ -tensor products under the assumption that the vertex operator algebra has the properties in Propositions 14.1 and 14.7. We have shown that these properties are in fact also necessary conditions for the existence of the associativity isomorphisms. In this section, we give some sufficient conditions for a vertex operator algebra to have the properties in Propositions 14.1 and 14.7. For concrete vertex operator algebras, for example, the vertex operator algebras associated to Wess-Zumino-Novikov-Witten models and minimal models, instead of proving the properties in Propositions 14.1 and 14.7, we can prove the sufficient conditions given in this section.

Let  $V$  be a rational vertex operator algebra having the property that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded. We see that any  $V$ -module must also be  $\mathbb{R}$ -graded, that is, the weight of an element in a  $V$ -module is always a real number.

Given any  $V$ -modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , let  $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$  and  $\mathcal{Y}_4$  be intertwining operators of type  $\begin{pmatrix} W_4 \\ W_1 W_5 \end{pmatrix}, \begin{pmatrix} W_5 \\ W_2 W_3 \end{pmatrix}, \begin{pmatrix} W_5 \\ W_1 W_2 \end{pmatrix}$  and  $\begin{pmatrix} W_4 \\ W_5 W_3 \end{pmatrix}$ , respectively. Consider the following conditions for the product of  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  and for the iterate of  $\mathcal{Y}_3$  and  $\mathcal{Y}_4$ , respectively:

**Convergence and extension property for products** There exists an integer  $N$  depending only on  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$ , and for any  $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3, w'_{(4)} \in W'_4$ , there exist  $j \in \mathbb{N}, r_i, s_i \in \mathbb{R}, i = 1, \dots, j$ , and analytic functions  $f_i(z)$  on  $|z| < 1, i = 1, \dots, j$ , satisfying

$$\text{wt } w_{(1)} + \text{wt } w_{(2)} + s_i > N, \quad i = 1, \dots, j, \quad (16.1)$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_2) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \quad (16.2)$$

is convergent when  $|z_1| > |z_2| > 0$  and can be analytically extended to the multi-valued analytic function

$$\sum_{i=1}^j z_2^{r_i} (z_1 - z_2)^{s_i} f_i \left( \frac{z_1 - z_2}{z_2} \right) \quad (16.3)$$

when  $|z_2| > |z_1 - z_2| > 0$ .

**Convergence and extension property for iterates** There exists an integer  $\tilde{N}$  depending only on  $\mathcal{Y}_3$  and  $\mathcal{Y}_4$ , and for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w'_{(4)} \in W'_4$ , there exist  $k \in \mathbb{N}$ ,  $\tilde{r}_i, \tilde{s}_i \in \mathbb{R}$ ,  $i = 1, \dots, k$ , and analytic functions  $\tilde{f}_i(z)$  on  $|z| < 1$ ,  $i = 1, \dots, k$ , satisfying

$$\text{wt } w_{(2)} + \text{wt } w_{(3)} + \tilde{s}_i > \tilde{N}, \quad i = 1, \dots, k, \quad (16.4)$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_4(\mathcal{Y}_3(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)} \rangle_{W_4} \Big|_{x_0=z_1-z_2, x_2=z_2} \quad (16.5)$$

is convergent when  $|z_2| > |z_1 - z_2| > 0$  and can be analytically extended to the multi-valued analytic function

$$\sum_{i=1}^k z_1^{\tilde{r}_i} z_2^{\tilde{s}_i} \tilde{f}_i \left( \frac{z_2}{z_1} \right) \quad (16.6)$$

when  $|z_1| > |z_2| > 0$ .

**Remark 16.1** If  $V$  is rational, we can always choose  $j$ ,  $r_i, s_i$ ,  $i = 1, \dots, j$ , and  $f_i(z)$ ,  $i = 1, \dots, j$ , such that

$$r_i + s_i = \Delta, \quad i = 1, \dots, j \quad (16.7)$$

where

$$\Delta = \text{wt } w_{(1)} + \text{wt } w_{(2)} + \text{wt } w_{(3)} - \text{wt } w'_{(4)}. \quad (16.8)$$

Similarly, for the convergence and extension property for iterates, we can always choose  $k$ ,  $\tilde{r}_i, \tilde{s}_i$ ,  $i = 1, \dots, k$ , and  $\tilde{f}_i$ ,  $i = 1, \dots, k$ , such that

$$\tilde{r}_i + \tilde{s}_i = \Delta, \quad i = 1, \dots, k. \quad (16.9)$$

If for any  $V$ -modules  $W_1, W_2, W_3, W_4$  and  $W_5$  and any intertwining operators  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  of the types as above, the convergence and extension property for products holds, we say that *the products of the intertwining operators for  $V$  have the convergence and extension property*. Similarly we can define what *the iterates of the intertwining operators for  $V$  have the convergence and extension property* means.

We also need the following concept: If a generalized  $V$ -module  $W = \coprod_{n \in \mathbb{C}} W_{(n)}$  satisfies the condition that  $W_{(n)} = 0$  for  $n$  whose real part is sufficiently small, we say that  $W$  is a *lower-truncated generalized  $V$ -module*.

The following theorem is the main result of this section:

**Theorem 16.2** *Assume that  $V$  is a rational vertex operator algebra and that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded. Also assume that  $V$  satisfies the following conditions:*

1. *Every finitely-generated lower-truncated generalized  $V$ -module is a  $V$ -module.*
2. *The products or the iterates of the intertwining operators for  $V$  have the convergence and extension property.*

*Then  $V$  has all the properties in Propositions 14.1 and 14.7.*

*Proof* By the convergence and extension property,  $V$  has the properties in Proposition 14.1. By Proposition 14.7, we need only to prove that  $V$  has the first property in Proposition 14.7, that is, for any  $V$ -modules  $W_1, W_2, W_3, W_4$  and  $W_5$ , any  $z_1$  and  $z_2$  satisfying (14.7) and any  $P(z_1)$ - and  $P(z_2)$ -intertwining maps  $F_1$  and  $F_2$  of the types as above, the product  $\gamma(F_1; I, F_2)$  of  $F_1$  and  $F_2$  satisfies the  $P(z_1 - z_2)$ -local grading-restriction condition.

By assumption, there exist  $j \in \mathbb{N}$ ,  $r_i, s_i \in \mathbb{C}$ ,  $i = 1, \dots, j$ , and analytic functions  $f_i(z)$  on  $|z| < 1$ ,  $i = 1, \dots, j$ , satisfying (16.1) and (16.7) such that when  $|z_1| > |z_2| > 0$ , (14.1), or equivalently (14.4), is absolutely convergent and it has an analytic extension in the region  $|z_2| > |z_1 - z_2| > 0$  of the form

$$\sum_{i=1}^j e^{r_i \log z_2} e^{s_i \log(z_1 - z_2)} f_i \left( \frac{z_1 - z_2}{z_2} \right) \quad (16.10)$$

for any homogeneous  $w'_{(4)} \in W'_4$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ . Expanding  $f_i$ ,  $i = 1, \dots, j$ , we can write (16.10) as

$$\sum_{i=1}^j \sum_{m \in \mathbb{N}} C_{im}(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) e^{(r_i-m) \log z_2} e^{(s_i+m) \log(z_1-z_2)}. \quad (16.11)$$

For any  $n \in \mathbb{C}$ , let

$$a_n(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) = \sum_{-r_i+m-1=n} C_{im}(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}). \quad (16.12)$$

Then (16.11) can be written as

$$\sum_{n \in \mathbb{C}} a_n(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) e^{(\Delta+n+1) \log(z_1-z_2)} e^{(-n-1) \log z_2}. \quad (16.13)$$

If  $n \in \mathbb{C}$  satisfies

$$n \neq -r_i + m - 1 = -\Delta + s_i + m - 1 \quad (16.14)$$

for any  $i$ ,  $1 \leq i \leq j$ , and any  $m \in \mathbb{N}$ , then by definition

$$a_n(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) = 0. \quad (16.15)$$

Since  $m \geq 0$ , by (16.1), (16.7) and (16.14) we see that for  $n \in \mathbb{C}$ , if

$$n + 1 + \text{wt } w'_{(4)} - \text{wt } w_{(3)} \leq N, \quad (16.16)$$

(16.15) holds.

When (14.7) holds,

$$\sum_{n \in \mathbb{C}} a_n(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) e^{(-\Delta+n+1) \log(z_1-z_2)} e^{(-n-1) \log z_2} \quad (16.17)$$

converges absolutely to (14.1) or (14.4). From the definition, we see that the series (16.17) is of the form (14.50) as a function of  $z_2$ . For  $w'_{(4)} \in W'_4$  and  $w_{(3)} \in W_3$ , let  $\beta_n(w'_{(4)}, w_{(3)}) \in (W_1 \otimes W_2)^*$  be defined by

$$\beta_n(w'_{(4)}, w_{(3)})(w_{(1)} \otimes w_{(2)}) = a_n(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) \quad (16.18)$$

for all  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ . By definition the series

$$\sum_{\Re(n) \in \mathbb{C}} \beta_n(w'_{(4)}, w_{(3)}) e^{(\Delta+n+1) \log(z_1-z_2)} e^{(-n-1) \log z_2}$$

is absolutely convergent to  $\mu_{\gamma(F_1; I, F_2)'(w'_{(4)}), w_{(3)}}^{(2)}$  and is indexed by sequences of strictly increasing real numbers. To show that  $\gamma(F_1; I, F_2)'(w'_{(4)})$  satisfies the  $P(z_1 - z_2)$ -local grading-restriction condition, we need to calculate

$$(v_1)_{m_1} \cdots (v_r)_{m_r} \beta_n(w'_{(4)}, w_{(3)})$$

and its weight for any  $r \in \mathbb{N}$ ,  $v_1, \dots, v_r \in V$ ,  $m_1, \dots, m_r \in \mathbb{Z}$ ,  $n \in \mathbb{C}$ , where  $(v_1)_{m_1}, \dots, (v_r)_{m_r}$ ,  $m_1, \dots, m_r \in \mathbb{Z}$ , are the components of  $Y'_{P(z_1 - z_2)}(v_1, x)$ ,  $\dots$ ,  $Y'_{P(z_1 - z_2)}(v_r, x)$ , respectively, on  $(W_1 \otimes W_2)^*$ . For convenience, we instead calculate

$$(v_1^*)_{m_1} \cdots (v_r^*)_{m_r} \beta_n(w'_{(4)}, w_{(3)}),$$

where  $(v_1^*)_{m_1}, \dots, (v_r^*)_{m_r}$ ,  $m_1, \dots, m_r \in \mathbb{Z}$ , are the components of the opposite vertex operators  $Y_{P(z_1 - z_2)}'^*(v_1, x)$ ,  $\dots$ ,  $Y_{P(z_1 - z_2)}'^*(v_r, x)$ , respectively. By the definition of  $Y_{P(z_1 - z_2)}'(v, x)$  and

$$Y_{P(z_1 - z_2)}'^*(v, x) = Y_{P(z_1 - z_2)}'(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}),$$

we have

$$\begin{aligned} & (Y_{P(z_1 - z_2)}'^*(v, x) \beta_n(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) = \\ &= \text{Res}_y (z_1 - z_2)^{-1} \delta \left( \frac{x - y}{z_1 - z_2} \right) (\beta_n(w'_{(4)}, w_{(3)}))(Y_1(v, y)w_{(1)} \otimes w_{(2)}) \\ & \quad + (\beta_n(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes Y_2(v, x)w_{(2)}) \\ &= \text{Res}_y (z_1 - z_2)^{-1} \delta \left( \frac{x - y}{z_1 - z_2} \right) a_n(w'_{(4)}, Y_1(v, y)w_{(1)}, w_{(2)}, w_{(3)}) \\ & \quad + a_n(w'_{(4)}, w_{(1)}, Y_2(v, x)w_{(2)}, w_{(3)}). \end{aligned} \tag{16.19}$$

On the other hand, since (16.17) is equal to (14.4) when (14.7) holds, we have

$$\begin{aligned} & \text{Res}_y (z_1 - z_2)^{-1} \delta \left( \frac{x - y}{z_1 - z_2} \right) \sum_{r \in \mathbb{C}} a_r(w'_{(4)}, Y_1(v, y)w_{(1)}, w_{(2)}, w_{(3)}) \cdot \\ & \quad \cdot e^{(\Delta + r + 1) \log(z_1 - z_2)} e^{(-r - 1) \log z_2} \\ & \quad + \sum_{r \in \mathbb{C}} a_r(w'_{(4)}, w_{(1)}, Y_1(v, y)w_{(2)}, w_{(3)}) e^{(\Delta + r + 1) \log(z_1 - z_2)} e^{(-r - 1) \log z_2} \\ &= \text{Res}_y (z_1 - z_2)^{-1} \delta \left( \frac{x - y}{z_1 - z_2} \right) \langle w'_{(4)}, \mathcal{Y}_1(Y_1(v, y)w_{(1)}, x_1) \cdot \end{aligned}$$

$$\begin{aligned}
& \cdot \mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
& + \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(Y_2(v, x)w_{(2)}, x_2)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
= & \text{Res}_y (x_1 - x_2)^{-1} \delta \left( \frac{x - y}{x_1 - x_2} \right) \langle w'_{(4)}, \mathcal{Y}_1(Y_1(v, y)w_{(1)}, x_1) \cdot \\
& \cdot \mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
& + \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(Y_2(v, x)w_{(2)}, x_2)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} . \quad (16.20)
\end{aligned}$$

Using the Jacobi identity for the intertwining operators  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$ , the right-hand side of (16.20) is equal to

$$\begin{aligned}
& \langle w'_{(4)}, Y_1(v, x - x_2)\mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
& - \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)Y_5(v, x - x_2)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
& + \text{Res}_y x^{-1} \delta \left( \frac{y - x_2}{x} \right) \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)Y_5(v, y) \cdot \\
& \cdot \mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
& - \text{Res}_y x^{-1} \delta \left( \frac{x_2 - y}{x} \right) \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \cdot \mathcal{Y}_2(w_{(2)}, x_2)Y_3(v, y)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
= & \text{Res}_y x^{-1} \delta \left( \frac{y - x_2}{x} \right) \langle w'_{(4)}, Y_4(v, y)\mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \cdot \mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
& - \text{Res}_y x^{-1} \delta \left( \frac{x_2 - y}{x} \right) \text{Res}_{x_2} x_2^n \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \cdot \mathcal{Y}_2(w_{(2)}, x_2)Y_2(v, y)w_{(3)}\rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} . \quad (16.21)
\end{aligned}$$

Thus

$$\text{Res}_y (z_1 - z_2)^{-1} \delta \left( \frac{x - y}{z_1 - z_2} \right) .$$

$$\begin{aligned}
& \cdot \sum_{r \in \mathbb{C}} a_r(w'_{(4)}, Y_1(v, y)w_{(1)}, w_{(2)}, w_{(3)}) e^{(\Delta+r+1) \log(z_1-z_2)} e^{(-r-1) \log z_2} \\
& + \sum_{r \in \mathbb{C}} a_r(w'_{(4)}, w_{(1)}, Y_1(v, y)w_{(2)}, w_{(3)}) \cdot \\
& \quad \cdot e^{(\Delta+r+1) \log(z_1-z_2)} e^{(-r-1) \log z_2} \\
& = \text{Res}_y x^{-1} \delta \left( \frac{y-x_2}{x} \right) \langle w'_{(4)}, Y(v, y) \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \quad \cdot \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
& - \text{Res}_y x^{-1} \delta \left( \frac{x_2-y}{x} \right) \text{Res}_{x_2} x_2^n \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \quad \cdot \mathcal{Y}_2(w_{(2)}, x_2) Y(v, y) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
& = \sum_{m \in \mathbb{Z}} \sum_{l \geq 0} (-1)^l \binom{m}{l} x^{-m-1} x_2^l \langle w_{(4)}, v_{m-l} \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \quad \cdot \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
& - \sum_{m \in \mathbb{Z}} \sum_{l \geq 0} (-1)^{l+m} \binom{m}{l} x^{-m-1} x_2^{m-l} \langle w_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \cdot \\
& \quad \cdot \mathcal{Y}_2(w_{(2)}, x_2) v_l w_{(3)} \rangle_{W_4} \Big|_{x_1=z_1, x_2=z_2} \\
& = \sum_{m \in \mathbb{Z}} \sum_{l \geq 0} (-1)^l \binom{m}{l} x^{-m-1} \sum_{r \in \mathbb{C}} a_r(v_{m-l}^* w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) \cdot \\
& \quad \cdot e^{(\Delta+r+1) \log(z_1-z_2)} e^{(l-r-1) \log z_2} \\
& - \sum_{m \in \mathbb{Z}} \sum_{l \geq 0} (-1)^{l+m} \binom{m}{l} x^{-m-1} \sum_{r \in \mathbb{C}} a_r(v_{m-l}^* w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) \cdot \\
& \quad \cdot e^{(\Delta+r+1) \log(z_1-z_2)} e^{(-m+l-r-1) \log z_2}. \tag{16.22}
\end{aligned}$$

The intermediate steps in the equality (16.20) holds only when (14.7) holds. But since both sides of (16.22) can be analytically extended to  $|z_2| > |z_1 - z_2| > 0$ , they are equal when  $|z_2| > |z_1 - z_2| > 0$ . By Lemma 14.5, the coefficients of both sides of (16.22) in powers of  $e^{\log z_2}$  are equal, that is,

$$\begin{aligned}
& \text{Res}_y (z_1 - z_2)^{-1} \delta \left( \frac{x-y}{z_1 - z_2} \right) \cdot a_n(w'_{(4)}, Y_1(v, y)w_{(1)}, w_{(2)}, w_{(3)}) \\
& + a_n(w'_{(4)}, w_{(1)}, Y_1(v, y)w_{(2)}, w_{(3)})
\end{aligned}$$



$$\begin{aligned}
&= \sum_{m \in \mathbb{Z}} \sum_{l \geq 0} (-1)^l \binom{m}{l} x^{-m-1} a_{n+l} (v_{m-l}^* w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) \\
&\quad - \sum_{m \in \mathbb{Z}} \sum_{l \geq 0} (-1)^{l+m} \binom{m}{l} x^{-m-1} a_{n+m-l} (v_{m-l}^* w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}).
\end{aligned} \tag{16.23}$$

By (16.17) and (16.23), we obtain

$$\begin{aligned}
&(v_m^* \beta_n(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) \\
&= \sum_{l \geq 0} (-1)^l \binom{m}{l} (\beta_{n+l}(v_{m-l}^* w'_{(4)}, w_{(3)}))(w_{(1)}, w_{(2)}) \\
&\quad - \sum_{l \geq 0} (-1)^{l+m} \binom{m}{l} (\beta_{n+m-l}(w'_{(4)}, v_l w_{(3)}))(w_{(1)}, w_{(2)}).
\end{aligned} \tag{16.24}$$

By induction,

$$\begin{aligned}
&((v_1^*)_{m_1} \cdots (v_r^*)_{m_r} \beta_n(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) = \\
&= \sum_{i \in \mathbb{N}} \sum_{\substack{j_1 > \cdots > j_i \\ j_{i+1} > \cdots > j_r \\ \{j_1, \dots, j_r\} = \{1, \dots, r\}}} \sum_{l_1, \dots, l_r \geq 0} (-1)^{l_1 + \cdots + l_r + (m_{j_{i+1}} + 1) + \cdots + (m_{j_r} + 1)} \cdot \\
&\quad \cdot \binom{m_{j_1}}{l_1} \cdots \binom{m_{j_r}}{l_r} (\beta_{n+m_{j_{i+1}} + \cdots + m_{j_r} + l_1 + \cdots + l_i - l_{i+1} - \cdots - l_r} \\
&\quad \cdot ((v_{j_1}^*)_{m_{j_1} - l_1} \cdots (v_{j_i}^*)_{m_{j_i} - l_i} w'_{(4)}, v_{l_{i+1}} \cdots v_{l_r} w_{(3)}))(w_{(1)} \otimes w_{(2)})
\end{aligned} \tag{16.25}$$

for any  $m_1, \dots, m_r \in \mathbb{Z}$  and any  $v_1, \dots, v_r \in V$ . Taking  $v = \omega$  and  $m = 1$  in (16.24), we have

$$\begin{aligned}
&(L'_{P(z_1 - z_2)}(0) \beta_n(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) \\
&= (\beta_n(L'(0)w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) \\
&\quad - (\beta_{n+1}(L'(1)w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) \\
&\quad - (\beta_n(w'_{(4)}, L(0)w_{(3)}))(w_{(1)} \otimes w_{(2)}) \\
&\quad + (\beta_{n+1}(w'_{(4)}, L(-1)w_{(3)}))(w_{(1)} \otimes w_{(2)}).
\end{aligned} \tag{16.26}$$

When  $|z_1| > |z_2| > |z_0| > 0$  where  $z_0 = z_1 - z_2$ ,

$$\begin{aligned}
& - \sum_{n \in \mathbb{C}} a_n(L'(1)w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) e^{(\Delta+n+1) \log(z_1-z_2)} e^{(-n-1) \log z_2} \\
& \quad + \sum_{n \in \mathbb{C}} a_n(w'_{(4)}, w_{(1)}, w_{(2)}, L(-1)w_{(3)}) e^{(\Delta+n+1) \log(z_1-z_2)} e^{(-n-1) \log z_2} \\
& = - \langle L'(1)w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_2+z_0, x_2=z_2} \\
& \quad + \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) L(-1)w_{(3)} \rangle_{W_4} \Big|_{x_1=z_2+z_0, x_2=z_2}. \quad (16.27)
\end{aligned}$$

By the commutator formula for  $L(-1)$  and intertwining operators and the  $L(-1)$ -derivative property for intertwining operators, the right-hand side of (16.27) is equal to

$$\begin{aligned}
& - \langle w'_{(4)}, \frac{d}{dx_1}(\mathcal{Y}_1(w_{(1)}, x_1)) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1=z_2+z_0, x_2=z_2} \\
& \quad - \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1) \frac{d}{dx_2}(\mathcal{Y}_2(w_{(2)}, x_2) w_{(3)}) \rangle_{W_4} \Big|_{x_1=z_2+z_0, x_2=z_2} \\
& = - \frac{\partial}{\partial z_2} \left( \langle w'_{(4)}, (\mathcal{Y}_1(w_{(1)}, x_1) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)}) \rangle_{W_4} \Big|_{x_1=z_2+z_0, x_2=z_2} \right) \\
& = - \frac{\partial}{\partial z_2} \left( \sum_{n \in \mathbb{C}} a_n(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) e^{(\Delta+n+1) \log z_0} e^{(-n-1) \log z_2} \right) \\
& = - \sum_{n \in \mathbb{C}} (-n-1) a_n(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) e^{(\Delta+n+1) \log z_0} e^{(-n-2) \log z_2}, \quad (16.28)
\end{aligned}$$

where  $\frac{\partial}{\partial z_2}$  is the partial derivation with respect to  $z_2$  acting on functions of  $z_0$  and  $z_2$ . By the calculations from (16.27) to (16.28), the left-hand side of (16.27) and the right-hand side of (16.28) are equal when (14.7) holds. Since they both can be extended to  $|z_2| > |z_1 - z_2| > 0$ , they are equal when  $|z_2| > |z_1 - z_2| > 0$ . By Lemma 14.5, their coefficients in powers of  $e^{\log z_2}$  are equal. Thus

$$\begin{aligned}
& -(\beta_{n+1}(L'(-1)w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) \\
& \quad + (\beta_{n+1}(w'_{(4)}, L(-1)w_{(3)}))(w_{(1)} \otimes w_{(2)}) \\
& = -a_{n+1}(L'(-1)w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) + a_{n+1}(w'_{(4)}, w_{(1)}, w_{(2)}, L(-1)w_{(3)})
\end{aligned}$$

$$\begin{aligned}
&= (n+1)a_n(w'_{(4)}, w_{(1)}, w_{(2)}, w_{(3)}) \\
&= (n+1)\beta_n(w'_{(4)}, w_{(3)})(w_{(1)} \otimes w_{(2)}).
\end{aligned} \tag{16.29}$$

Substituting (16.29) into (16.26), we have

$$\begin{aligned}
&(L'_{P(z_1-z_2)}(0)\beta_n(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) = \\
&= (\text{wt } w'_{(4)} + n + 1 - \text{wt } w_{(3)})(\beta_n(w'_{(4)}, w_{(3)}))(w_{(1)} \otimes w_{(2)}) \tag{16.30}
\end{aligned}$$

for any  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ . Thus  $\beta_n(w'_{(4)}, w_{(3)})$ ,  $n \in \mathbb{C}$ , are weight vectors when  $w_{(3)}$  and  $w'_{(4)}$  are homogeneous and their weights are  $\text{wt } w'_{(4)} + n + 1 - \text{wt } w_{(3)}$ . By (16.23) we see that  $(v_1^*)_{m_1} \cdots (v_r^*)_{m_r} \beta_n(w'_{(4)}, w_{(3)})$ ,  $n \in \mathbb{C}$ , are weight vectors when  $v_1, \dots, v_r, w_{(3)}$  and  $w'_{(4)}$  are homogeneous and their weights are

$$-(\text{wt } v_1 - m_1 - 1) - \cdots - (\text{wt } v_r - m_r - 1) + \text{wt } w'_{(4)} + n + 1 - \text{wt } w_{(3)}. \tag{16.31}$$

By the definition of  $\beta_n(w'_{(4)}, w_{(3)})$ ,  $\beta_n(w'_{(4)}, w_{(3)}) = 0$  when

$$\text{wt } \beta_n(w'_{(4)}, w_{(3)}) = \text{wt } w'_{(4)} + n + 1 - \text{wt } w_{(3)} \leq N.$$

Thus by (16.25)

$$\begin{aligned}
&(v_1^*)_{m_1} \cdots (v_r^*)_{m_r} \beta_n(w'_{(4)}, w_{(3)}) = 0 \\
&\text{when } \text{wt } (v_1^*)_{m_1} \cdots (v_r^*)_{m_r} \beta_n(w'_{(4)}, w_{(3)}) \leq N. \tag{16.32}
\end{aligned}$$

For fixed  $n \in \mathbb{C}$ ,  $w_{(3)} \in W_3$  and  $w'_{(4)} \in W'_4$ , let  $W_{\beta_n(w'_{(4)}, w_{(3)})}$  be the smallest graded space containing  $\beta_n(w'_{(4)}, w_{(3)})$  and stable under the action of  $Y'^*_{P(z_1-z_2)}$  or equivalently under  $Y'_{P(z_1-z_2)}$ . Then (16.32) shows that the homogeneous subspace  $(W_{\beta_n(w'_{(4)}, w_{(3)})})_{(l)}$  of a fixed weight  $l \in \mathbb{C}$  of  $W_{\beta_n(w'_{(4)}, w_{(3)})}$  is 0 when  $l$  is sufficiently small. Since  $\gamma(F_1; I, F_2)'(w'_{(4)})$  satisfies the  $P(z_1, z_2)$ -compatibility condition, by the proof of (14.51),  $\beta_n$  satisfies the  $P(z_1 - z_2)$ -compatibility condition. Thus  $W_{\beta_n(w'_{(4)}, w_{(3)})}$  is a finitely-generated lower-truncated generalized module. By assumption, it is in fact a module. This proves that for any  $n \in \mathbb{C}$ ,  $\beta_n(w'_{(4)}, w_{(3)})$  satisfies the  $P(z_1 - z_2)$ -local grading-restriction condition. So  $\beta_n(w'_{(4)}, w_{(3)})$  is an element of  $W_1 \mathfrak{N}_{P(z_1-z_2)} W_2$ . Since  $\beta_n(w'_{(4)}, w_{(3)})$ ,  $n \in \mathbb{C}$ , are all elements of  $W_1 \mathfrak{N}_{P(z_1-z_2)} W_2$ , it follows that  $\gamma(F_1; I, F_2)'(w'_{(4)})$  satisfies the  $P(z_1 - z_2)$ -local grading-restriction condition.  $\square$

Combining Theorems 14.10 and 16.2, we obtain:

**Theorem 16.3** *Assume that  $V$  is a rational vertex operator algebra and that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded. Also assume that  $V$  satisfies the following conditions:*

1. *Every finitely-generated lower-truncated generalized  $V$ -module is a  $V$ -module.*
2. *The products or the iterates of the intertwining operators for  $V$  have the convergence and extension property.*

*Then for any  $V$ -module  $W_1$ ,  $W_2$  and  $W_3$  and any complex numbers  $z_1$  and  $z_2$  satisfying (14.7), there is a unique isomorphism  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$  from  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$  to  $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$  such that (14.76) holds.  $\square$*

We now would like to know when every finitely-generated lower-truncated generalized  $V$ -module is a module. We consider the following conditions for  $V$ :

**Condition A** The vertex operator algebra  $V$  satisfies the following conditions:

1.  $V$  is finitely generated by  $v_1, \dots, v_k \in V$ .
2. For any  $i_1, \dots, i_m \in \mathbb{Z}$  satisfying  $1 \leq i_1, \dots, i_m \leq k$  and any  $n_1, \dots, n_m \in \mathbb{C}$ , the operator

$$(v_{i_1})_{n_1} \cdots (v_{i_m})_{n_m} \quad (16.33)$$

on  $V$  can be written as a linear combination of the operators

$$(v_{j_1})_{l_1} \cdots (v_{j_p})_{l_p}, \quad p > 0, \quad 1 \leq j_1, \dots, j_p \leq k, \quad l_1, \dots, l_p \in \mathbb{C}, \quad (16.34)$$

having the property that there exist integers  $p_1$  and  $p_2$  satisfying  $0 \leq p_1 \leq p_2 \leq p$  such that

$$\begin{aligned} \text{wt } v_{j_q} - l_q - 1 &< 0, & 1 \leq j \leq p_1 \\ \text{wt } v_{j_q} - l_q - 1 &> 0, & p_1 < j \leq p_2 \\ \text{wt } v_{j_q} - l_q - 1 &= 0, & p_2 < j \leq p. \end{aligned} \quad (16.35)$$

3. For any lower-truncated generalized  $V$ -module  $W$  generated by  $w \in W$ , the subspace of  $W$  spanned by the elements of the form

$$(v_{i_1})_{n_1} \cdots (v_{i_m})_{n_m} w, \quad \text{wt } v_{i_1} - n_1 - 1 = \cdots = \text{wt } v_{i_m} - n_m - 1 = 0 \quad (16.36)$$

is finite-dimensional.

**Proposition 16.4** *Let  $V$  be a vertex operator algebra satisfying Condition A. Then every finitely-generated lower-truncated generalized  $V$ -module  $W$  is a module.*

*Proof* We can assume that  $W$  is generated by one element  $w$ . It is clear from the Condition A that given any  $n \in \mathbb{C}$ , there are only finitely many elements in  $W_{(n)}$  of the form (16.34) satisfying (16.35). So  $W$  is a module.  $\square$

Combining this proposition with Theorem 16.3, we obtain:

**Theorem 16.5** *Let  $V$  be a rational vertex operator algebra. Assume that  $V$  satisfies the Condition A, that all irreducible  $V$ -modules are  $\mathbb{R}$ -graded and that the products or the iterates of the intertwining operators for  $V$  have the convergence and extension property. Then for any  $V$ -module  $W_1$ ,  $W_2$  and  $W_3$  and any complex numbers  $z_1$  and  $z_2$  satisfying (14.7), there is a unique isomorphism  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$  from  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$  to  $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$  such that (14.76) holds for any  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ .  $\square$*

## References

- [FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993.
- [HL1] Y.-Z. Huang and J. Lepowsky, Vertex operator algebras and operads, in: *The Gelfand Mathematical Seminars, 1990–1992*, ed. L. Corwin, I. Gelfand and J. Lepowsky, Birkhäuser, Boston, 1993, 145–161.

- [HL2] Y.-Z. Huang and J. Lepowsky, Operadic formulation of the notion of vertex operator algebra, in: *Proc. 1992 Joint Summer Research Conference on Conformal Field Theory, Topological Field Theory and Quantum Groups, Mount Holyoke, 1992*, Contemporary Math., Amer. Math. Soc., Providence, to appear.
- [HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica*, to appear.
- [HL4] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica*, to appear.
- [HL5] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebras and vertex tensor categories, in: *Lie Theory and Geometry, in honor of Bertram Kostant*, ed. R. Brylinski, J.-L. Brylinski, V. Guillemin, V. Kac, Birkhäuser, Boston, 1994, 349–383.
- [HL6] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.*, to appear.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104

and

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ 08903 (*current address*)

*E-mail address:* yzhuang@math.rutgers.edu